A SCHOOL GEOMETRY

PARTS I-VI

by

H. S. HALL, M.A.

and

F. H. STEVENS, M.A.

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1504

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PARTS I—VI.

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(Containing Plane and Solid Geometry, treated both theoretically and graphically)

BY

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PREFACE.

THE present work provides a course of Elementary Geometry based on the recommendations of the Mathematical Association and on the schedule recently proposed and adopted at Cambridge.

The principles which governed these proposals have been confirmed by the issue of revised schedules for all the more important Examinations, and they are now so generally accepted by teachers that they need no discussion here. It is enough to note the following points :

(i) We agree that a pupil should gain his first geometrical ideas from a short preliminary course of a practical and experimental character. A suitable introduction to the present book would consist of Easy Exercises in Drawing to illustrate the subject matter of the Definitions ; Measurements of Lines and Angles ; Use of Compasses and Protractor ; Problems on Bisection, Perpendiculars, and Parallels ; Use of Set Squares ; The Construction of Triangles and Quadrilaterals. These problems should be accompanied by informal explanation, and the results verified by measurement. Concurrently, there should be a series of exercises in Drawing and Measurement designed to lead inductively to the more important Theorems of Part I. [Euc. I. 1–34].* While strongly advocating some such introductory lessons, we may point out that our book, as far as it goes, is complete in itself, and from the first is illustrated by numerical and graphical examples of the easiest types. Thus, throughout the whole work, a graphical and experimental course is provided side by side with the usual deductive exercises.

(ii) Theorems and Problems are arranged in separate but parallel courses, intended to be studied *pari passu*. This arrangement is made possible by the use, now generally sanctioned, of *Hypothetical Constructions*. These, before being employed in the text, are carefully specified, and referred to the Axioms on which they depend.

* Such an introductory course is now furnished by our ***Lessons in Experimental and Practical Geometry.***

(iii) The subject is placed on the basis of *Commensurable Magnitudes.* By this means, certain difficulties which are wholly beyond the grasp of a young learner are postponed, and a wide field of graphical and numerical illustration is opened. Moreover the fundamental Theorems on Areas (hardly less than those on Proportion) may thus be reduced in number, greatly simplified, and brought into line with practical applications.

(iv) An attempt has been made to curtail the excessive body of text which the demands of Examinations have hitherto forced as "bookwork" on a beginner's memory. Even of the Theorems here given a certain number (which we have distinguished with an asterisk) might be omitted or postponed at the discretion of the teacher. And the formal propositions for which—as such—teacher and pupil are held responsible, might perhaps be still further limited to those which make the landmarks of Elementary Geometry. Time so gained should be used in getting the pupil to *apply* his knowledge; and the working of examples should be made as important a part of a lesson in Geometry as it is so considered in Arithmetic and Algebra.

Though we have not always followed Euclid's order of Propositions, we think it desirable for the present, in regard to the subject-matter of Euclid Book I., to preserve the essentials of his logical sequence. Our departure from Euclid's treatment of Areas has already been mentioned; the only other important divergence in this section of the work is the position of I. 26 (Theorem 17), which we place after I. 32 (Theorem 16), thus getting rid of the tedious and uninstructive *Second Case.* In subsequent Parts a freer treatment in respect of logical order has been followed.

As regards the presentment of the propositions, we have constantly kept in mind the needs of that large class of students, who, without special aptitude for mathematical study, and under no necessity for acquiring technical knowledge, may and do derive real intellectual advantage from lessons in pure deductive reasoning. Nothing has as yet been devised as effective for this purpose as the Euclidean form of proof; and in our opinion no excuse is needed for treating the earlier proofs with that fulness which we have always found necessary in our experience as teachers.

The examples are numerous and for the most part easy. They have been very carefully arranged, and are distributed throughout the text in immediate connection with the propositions on which they depend. A special feature is the large number of examples involving graphical or numerical work. The answers to these have been printed on perforated pages, so that they may easily be removed if it is found that access to numerical results is a source of temptation in examples involving measurement.

We are indebted to several friends for advice and suggestions. In particular we wish to express our thanks to Mr. H. C. Playne and Mr. H. C. Beaven of Clifton College for the valuable assistance they have rendered in reading the proof sheets and checking the answers to some of the numerical exercises.

H. S. HALL.
F. H. STEVENS.

November, 1903.

PREFATORY NOTE TO THE THIRD EDITION.

In the present edition some further steps have been taken towards the curtailment of bookwork by reducing certain less important propositions (e.g. Euclid I. 22, 43, 44) to the rank of exercises. Room has thus been found for more numerical and graphical exercises, and experimental work such as that leading to the Theorem of Pythagoras.

Theorem 22 (page 62), in the shape recommended in the Cambridge Schedule, replaces the equivalent proposition given as *Additional Theorem A* (page 60) in previous editions.

In the case of a few problems (e.g. Problems 23, 28, 29) it has been thought more instructive to justify the construction by a preliminary *analysis* than by the usual formal proof.

H. S. HALL.
F. H. STEVENS.

March, 1904.

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Lines and Planes.

7. When one straight line stands on another so as to make the adjacent angles equal to one another, each of the angles is called **a right angle**; and each line is said to be **perpendicular** to the other.

AXIOMS. (i) *If* O *is a point in a straight line* AB, *then a line* OC, *which turns about* O *from the position* OA *to the position* OB, *must pass through one position, and only one, in which it is perpendicular to* AB.

(ii) *All right angles are equal.*

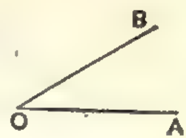
A right angle is divided into 90 equal parts called **degrees** (°); each degree into 60 equal parts called **minutes** (′); each minute into 60 equal parts called **seconds** (″).

In the above figure, if OC revolves about O from the position OA into the position OB, it turns through *two right angles*, or 180°.

If OC makes a *complete revolution* about O, starting from OA and returning to its original position, it turns through *four right angles*, or 360°.

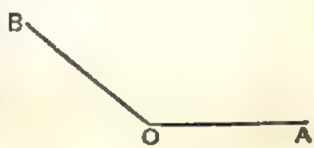
8. An angle which is less than one right angle is said to be **acute**.

That is, an acute angle is less than 90°.



9. An angle which is greater than *one* right angle, but less than *two* right angles, is said to be **obtuse**.

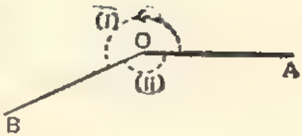
That is, an obtuse angle lies between 90° and 180°.



10. If one arm OB of an angle turns until it makes *a straight line* with the other arm OA, the angle so formed is called **a straight angle**.

A straight angle = 2 right angles = 180°.

11. An angle which is greater than *two* right angles, but less than *four* right angles, is said to be reflex.

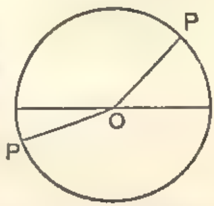


That is, a reflex angle lies between 180° and 360°.

NOTE. When two straight lines meet, *two* angles are formed, one greater, and one less than two right angles. The first arises by supposing OB to have revolved from the position OA the longer way round, marked (i); the other by supposing OB to have revolved the shorter way round, marked (ii). Unless the contrary is stated, the angle between two straight lines will be considered to be that which is less than two right angles.

12. Any portion of a plane surface bounded by one or more lines is called a plane figure.

13. A circle is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same.



Here the point P moves so that its distance from the fixed point O is always the same.

The fixed point is called the centre, and the bounding line is called the circumference.

14. A radius of a circle is a straight line drawn from the centre to the circumference. It follows that all radii of a circle are equal.

15. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

16. An arc of a circle is any part of the circumference.

GEOMETRY.

PART I.

Axioms.

All mathematical reasoning is founded on certain simple principles, the truth of which is so evident that they are accepted without proof. These self-evident truths are called **Axioms.**

For instance:

Things which are equal to the same thing are equal to one another.

The following axioms, corresponding to the first four Rules of Arithmetic, are among those most commonly used in geometrical reasoning.

Addition. *If equals are added to equals, the sums are equal.*

Subtraction. *If equals are taken from equals, the remainders are equal.*

Multiplication. *Things which are the same multiples of equals are equal to one another.*

For instance: *Doubles of equals are equal to one another.*

Division. *Things which are the same parts of equals are equal to one another.*

For instance: *Halves of equals are equal to one another.*

The above Axioms are given as instances, and not as a complete list, of those which will be used. They are said to be *general*, because they apply equally to magnitudes of all kinds. Certain special axioms relating to geometrical magnitudes only will be stated from time to time as they are required.

Definitions and First Principles.

Every beginner knows in a general way what is meant by a point, a line, and a surface. But in geometry these terms are used in a strict sense which needs some explanation.

1. A point has position, but is said to have *no magnitude.*

This means that we are to attach to a point no idea of size either as to *length* or *breadth*, but to think only where it is situated. A dot made with a sharp pencil may be taken as roughly representing a point; but small as such a dot may be, it still has *some* length and breadth, and is therefore not actually a geometrical point. The smaller the dot however, the more nearly it represents a point.

2. A line has length, but is said to have *no breadth.*

A line is traced out by a moving point. If the point of a pencil is moved over a sheet of paper, the trace left represents a line. But such a trace, however finely drawn, has some degree of breadth, and is therefore not itself a true geometrical line. The finer the trace left by the moving pencil-point, the more nearly will it represent a line.

3. Proceeding in a similar manner from the idea of a line to the idea of a surface, we say that

A surface has length and breadth, but *no thickness.*

And finally,

A solid has length, breadth, and thickness.

Solids, surfaces, lines and points are thus related to one another:

(i) A solid is bounded by surfaces.

(ii) A surface is bounded by lines; and surfaces meet in lines.

(iii) A line is bounded (or terminated) by points; and lines meet in points.

4. A line may be straight or curved.

A straight line has the same direction from point to point throughout its whole length.

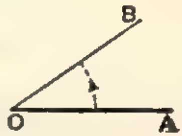
A curved line changes its direction continually from point to point.

AXIOM. *There can be only one straight line joining two given points:* that is,

Two straight lines cannot enclose a space.

5. A plane is a *flat* surface, the test of flatness being that if any two points are taken in the surface, the straight line between them lies wholly in that surface.

6. When two straight lines meet at a point, they are said to form an angle.



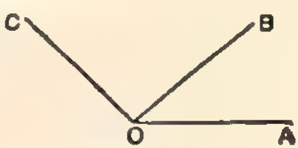
The straight lines are called the arms of the angle; the point at which they meet is its vertex.

The magnitude of the angle may be thus explained:

Suppose that the arm OA is fixed, and that OB *turns about the point* O (as shewn by the arrow). Suppose also that OB began its turning from the position OA. Then the size of the angle AOB is measured by the *amount of turning* required to bring the revolving arm from its first position OA into its subsequent position OB.

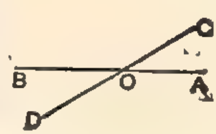
Observe that the size of an angle does not in any way depend on the length of its arms.

Angles which lie on either side of a common arm are said to be adjacent.



For example, the angles AOB, BOC, which have the common arm OB, are adjacent.

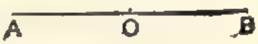
When two straight lines such as AB, CD cross one another at O, the angles COA, BOD are said to be vertically opposite. The angles AOD, COB are also vertically opposite to one another.



17. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

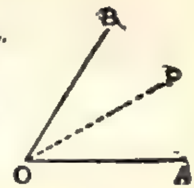
18. To bisect means to divide into two *equal* parts.

AXIOMS. (i) *If a point* O *moves from* A *to* B *along the straight line* AB, *it must pass through one position in which it divides* AB *into two equal parts.*



That is to say:
Every finite straight line has a point of bisection.

(ii) *If a line* OP, *revolving about* O, *turns from* OA *to* OB, *it must pass through one position in which it divides the angle* AOB *into two equal parts.*



That is to say:
Every angle may be supposed to have a line of bisection.

HYPOTHETICAL CONSTRUCTIONS.

From the Axioms attached to Definitions 7 and 18, it follows that we may suppose

(i) *A straight line to be drawn perpendicular to a given straight line from any point in it.*

(ii) *A finite straight line to be bisected at a point.*

(iii) *An angle to be bisected by a line.*

SUPERPOSITION AND EQUALITY.

AXIOM. *Magnitudes which can be made to coincide with one another are equal.*

This axiom implies that any line, angle, or figure, may be taken up from its position, and without change in size or form, laid down upon a second line, angle, or figure, for the purpose of comparison, and it states that two such magnitudes are equal when one can be exactly placed over the other without overlapping.

This process is called **superposition**, and the first magnitude is said to be applied to the other.

Postulates.

In order to draw geometrical figures certain instruments are required. These are, for the purposes of this book, (i) *a straight ruler*, (ii) *a pair of compasses.* The following Postulates (or requests) claim the use of these instruments, and assume that with their help the processes mentioned below may be duly performed.

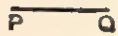
Let it be granted:

1. *That a straight line may be drawn from any one point to any other point.*

2. *That a* FINITE *(or terminated) straight line may be* PRODUCED *(that is, prolonged) to any length in that straight line.*

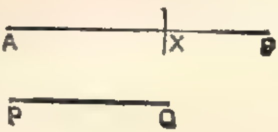
3. *That a circle may be drawn with any point as centre and with a radius of any length.*

NOTES. (i) Postulate 3, as stated above, implies that we may adjust the compasses to the length of any straight line PQ, and with a radius of this length draw a circle with any point O as centre. That is to say, the compasses may be used to *transfer distances* from one part of a diagram to another.



(ii) *Hence from* AB, *the greater of two straight lines, we may cut off a part equal to* PQ *the less.*

For if with centre A, and radius equal to PQ, we draw an arc of a circle cutting AB at X, it is obvious that AX is equal to PQ.



INTRODUCTORY.

1. Plane geometry deals with the properties of such lines and figures as may be drawn on a plane surface.

2. The subject is divided into a number of separate discussions, called **propositions**.

Propositions are of two kinds, **Theorems** and **Problems**.

A **Theorem** proposes to prove the truth of some geometrical statement.

A **Problem** proposes to perform some geometrical construction, such as to draw some particular line, or to construct some required figure.

3. A Proposition consists of the following parts:

The *General Enunciation*, the *Particular Enunciation*, the *Construction*, and the *Proof*.

(i) The **General Enunciation** is a preliminary statement, describing in general terms the purpose of the proposition.

(ii) The **Particular Enunciation** repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily.

(iii) The **Construction** then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem.

(iv) The **Proof** shews that the object proposed in a problem has been accomplished, or that the property stated in a theorem is true.

4. The letters Q.E.D. are appended to a theorem, and stand for **Quod erat Demonstrandum**, *which was to be proved.*

5. **A Corollary** is a statement the truth of which follows readily from an established proposition; it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof.

6. The following symbols and abbreviations are used in the text of this book.

In Part I.

$\therefore$	*for*	therefore,	$\angle$	*for*	angle,
$=$	„	is, or are, equal to,	$\triangle$	„	triangle.

After Part I.

pt.	*for*	point,	perp.	*for*	perpendicular,
st. line	„	straight line,	par^{m}	„	parallelogram,
rt. $\angle$	„	right angle,	rectil.	„	rectilineal,
par^{l} (or $\parallel$)	„	parallel,	$\odot$	„	circle,
sq.	„	square,	$\bigcirc^{ce}$	„	circumference;

and all obvious contractions of commonly occurring words, such as opp., adj., diag., etc., *for* opposite, adjacent, diagonal, etc.

[For convenience of oral work, and to prevent the rather common abuse of contractions by beginners, the above code of signs has been introduced gradually, and at first somewhat sparingly.]

In numerical examples the following abbreviations will be used.

m.	*for*	metre,	cm.	*for*	centimetre,
mm.	„	millimetre.	km.	„	kilometre.

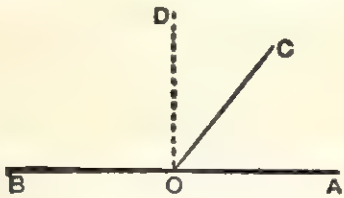
Also inches are denoted by the symbol (″).

Thus 5″ means 5 inches.

ON LINES AND ANGLES.

THEOREM 1. [Euclid I. 13.]

The adjacent angles which one straight line makes with another straight line on one side of it, are together equal to two right angles.



Let the straight line CO make with the straight line AB the adjacent ∠ˢAOC, COB.

It is required to prove that the ∠ˢAOC, COB *are together equal to two right angles.*

Suppose OD is at right angles to BA.

Proof. Then the ∠ˢAOC, COB together
= the three ∠ˢAOC, COD, DOB.

Also the ∠ˢAOD, DOB together
= the three ∠ˢAOC, COD, DOB.

∴ the ∠ˢAOC, COB together = the ∠ˢAOD, DOB
= two right angles.

Q.E.D.

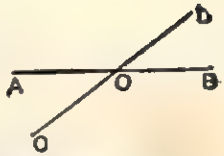
PROOF BY ROTATION.

Suppose a straight line revolving about O turns from the position OA into the position OC, and thence into the position OB; that is, let the revolving line turn in succession through the ∠ˢAOC, COB.

Now in passing from its first position OA to its final position OB, the revolving line turns through two right angles, for AOB is a straight line.

Hence the ∠ˢAOC, COB together = two right angles.

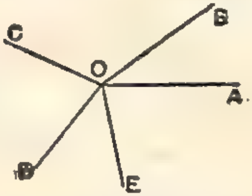
COROLLARY 1. *If two straight lines cut one another, the four angles so formed are together equal to four right angles.*



For example,

$$\angle BOD + \angle DOA + \angle AOC + \angle COB = 4 \text{ right angles.}$$

COROLLARY 2. *When any number of straight lines meet at a point, the sum of the consecutive angles so formed is equal to four right angles.*



For a straight line revolving about O, and turning in succession through the $\angle^s$ AOB, BOC, COD, DOE, EOA, will have made one complete revolution, and therefore turned through four right angles.

DEFINITIONS.

(i) Two angles whose sum is *two* right angles, are said to be supplementary; and each is called the supplement of the other.

Thus in the Fig. of Theor. 1 the angles AOC, COB are supplementary. Again the angle 123° is the supplement of the angle 57°.

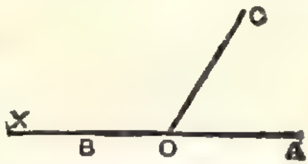
(ii) Two angles whose sum is *one* right angle are said to be complementary; and each is called the complement of the other.

Thus in the Fig. of Theor. 1 the angle DOC is the complement of the angle AOC. Again angles of 34° and 56° are complementary.

COROLLARY 3. (i) *Supplements of the same angle are equal.*
(ii) *Complements of the same angle are equal.*

THEOREM 2. [Euclid I. 14.]

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line.



At O in the straight line CO let the two straight lines OA, OB, on opposite sides of CO, make the adjacent ∠ˢ AOC, COB together equal to two right angles: (that is, let the adjacent ∠ˢ AOC, COB be supplementary).

It is required to prove that OB *and* OA *are in the same straight line.*

Produce AO beyond O to any point X: it will be shewn that OX and OB are the same line.

Proof. Since by construction AOX is a straight line,
∴ the ∠COX is the supplement of the ∠COA. *Theor.* 1.

But, by hypothesis,
the ∠COB is the supplement of the ∠COA.

∴ the ∠COX = the ∠COB;
∴ OX and OB are the same line.

But, by construction, OX is in the same straight line with OA;
hence OB is also in the same straight line with OA.

Q.E.D.

EXERCISES.

1. Write down the supplements of *one-half* of a right angle, *four-thirds* of a right angle; also of 46°, 149°, 83°, 101° 15′.

2. Write down the complement of *two-fifths* of a right angle; also of 27°, 38° 16′, and 41° 29′ 30″.

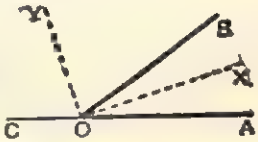
3. If two straight lines intersect forming four angles of which one is known to be a right angle, prove that the other three are also right angles.

4. In the triangle ABC the angles ABC, ACB are given equal. If the side BC is produced both ways, shew that the exterior angles so formed are equal.

5. In the triangle ABC the angles ABC, ACB are given equal. If AB and AC are produced beyond the base, shew that the exterior angles so formed are equal.

DEFINITION. The lines which bisect an angle and the adjacent angle made by producing one of its arms are called the **internal** and **external bisectors** of the given angle.

Thus in the diagram, OX and OY are the internal and external bisectors of the angle AOB.



6. Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle. That is to say, *the internal and external bisectors of an angle are at right angles to one another.*

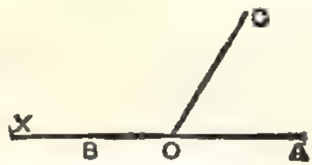
7. Shew that the angles AOX and COY in the above diagram are complementary.

8. Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

9. If the angle AOB is 35°, find the angle COY.

THEOREM 2. [Euclid I. 14.]

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line.



At O in the straight line CO let the two straight lines OA, OB, on opposite sides of CO, make the adjacent ∠ˢ AOC, COB together equal to two right angles: (that is, let the adjacent ∠ˢ AOC, COB be supplementary).

It is required to prove that OB *and* OA *are in the same straight line.*

Produce AO beyond O to any point X: it will be shewn that OX and OB are the same line.

Proof. Since by construction AOX is a straight line,
∴ the ∠COX is the supplement of the ∠COA. *Theor.* 1.

But, by hypothesis,
the ∠COB is the supplement of the ∠COA.

∴ the ∠COX = the ∠COB;
∴ OX and OB are the same line.

But, by construction, OX is in the same straight line with OA;
hence OB is also in the same straight line with OA.

Q.E.D.

EXERCISES.

1. Write down the supplements of *one-half* of a right angle, *four-thirds* of a right angle; also of 46°, 149°, 83°, 101° 15′.

2. Write down the complement of *two-fifths* of a right angle; also of 27°, 38° 16′, and 41° 29′ 30″.

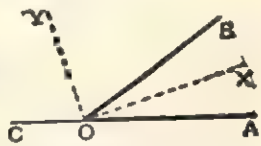
3. If two straight lines intersect forming four angles of which one is known to be a right angle, prove that the other three are also right angles.

4. In the triangle ABC the angles ABC, ACB are given equal. If the side BC is produced both ways, shew that the exterior angles so formed are equal.

5. In the triangle ABC the angles ABC, ACB are given equal. If AB and AC are produced beyond the base, shew that the exterior angles so formed are equal.

DEFINITION. The lines which bisect an angle and the adjacent angle made by producing one of its arms are called the **internal** and **external bisectors** of the given angle.

Thus in the diagram, OX and OY are the internal and external bisectors of the angle AOB.



6. Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle. That is to say, *the internal and external bisectors of an angle are at right angles to one another.*

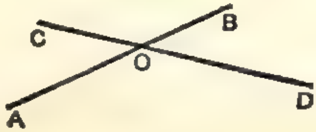
7. Shew that the angles AOX and COY in the above diagram are complementary.

8. Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

9. If the angle AOB is 35°, find the angle COY.

THEOREM 3. [Euclid I. 15.]

If two straight lines cut one another, the vertically opposite angles are equal.



Let the straight lines AB, CD cut one another at the point O.

It is required to prove that

(i) *the* $\angle$ AOC = *the* $\angle$ DOB;

(ii) *the* $\angle$ COB = *the* $\angle$ AOD.

Proof. Because AO meets the straight line CD,

$\therefore$ the adjacent $\angle$s AOC, AOD together = two right angles;
that is, the $\angle$ AOC is the supplement of the $\angle$ AOD.

Again, because DO meets the straight line AB,

$\therefore$ the adjacent $\angle$s DOB, AOD together = two right angles;
that is, the $\angle$ DOB is the supplement of the $\angle$ AOD.

Thus each of the $\angle$s AOC, DOB is the supplement of the $\angle$ AOD,

$\therefore$ the $\angle$ AOC = the $\angle$ DOB.

Similarly, the $\angle$ COB = the $\angle$ AOD.

Q.E.D.

PROOF BY ROTATION.

Suppose the line COD to revolve about O until OC turns into the position OA. Then at the same moment OD must reach the position OB (for AOB and COD are *straight*).

Thus the same amount of turning is required to close the $\angle$ AOC as to close the $\angle$ DOB.

$\therefore$ the $\angle$ AOC = the $\angle$ DOB.

EXERCISES ON ANGLES.

(Numerical.)

1. Through what angles does the minute-hand of a clock turn in (i) 5 minutes, (ii) 21 minutes, (iii) $43\frac{1}{2}$ minutes, (iv) 14 min. 10 sec.? And how long will it take to turn through (v) 66°, (vi) 222°?

2. A clock is started at noon: through what angles will the hour-hand have turned by (i) 3.45, (ii) 10 minutes past 5? And what will be the time when it has turned through $172\frac{1}{2}$°?

3. The earth makes a complete revolution about its axis in 24 hours. Through what angle will it turn in 3 hrs. 20 min., and how long will it take to turn through 130°?

4. In the diagram of Theorem 3

(i) If the $\angle$AOC$=35°$, write down (without measurement) the value of each of the $\angle$s COB, BOD, DOA.

(ii) If the $\angle$s COB, AOD together make up 250°, find each of the $\angle$s COA, BOD.

(iii) If the $\angle$s AOC, COB, BOD together make up 274°, find each of the four angles at O.

(Theoretical.)

5. If from O a point in AB two straight lines OC, OD are drawn on opposite sides of AB so as to make the angle COB equal to the angle AOD; shew that OC and OD are in the same straight line.

6. Two straight lines AB, CD cross at O. If OX is the bisector of the angle BOD, prove that XO produced bisects the angle AOC.

7. Two straight lines AB, CD cross at O. If the angle BOD is bisected by OX, and AOC by OY, prove that OX, OY are in the same straight line.

8. If OX bisects an angle AOB, shew that, by folding the diagram about the bisector, OA may be made to coincide with OB.

How would OA fall with regard to OB, if

(i) the $\angle$AOX were greater than the $\angle$XOB;

(ii) the $\angle$AOX were less than the $\angle$XOB?

9. AB and CD are straight lines intersecting at right angles at O; shew by folding the figure about AB, that OC may be made to fall along OD.

10. A straight line AOB is drawn on paper, which is then folded about O, so as to make OA fall along OB; shew that the crease left in the paper is perpendicular to AB.

ON TRIANGLES.

1. Any portion of a plane surface bounded by one or more lines is called a plane figure.

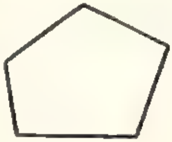
The sum of the bounding lines is called the perimeter of the figure. The amount of surface enclosed by the perimeter is called the area.

2. Rectilineal figures are those which are bounded by straight lines.

3. A triangle is a plane figure bounded by *three* straight lines.

4. A quadrilateral is a plane figure bounded by *four* straight lines.

5. A polygon is a plane figure bounded by more than four straight lines.



6. A rectilineal figure is said to be
   equilateral, when all its sides are equal;
   equiangular, when all its angles are equal;
   regular, when it is both equilateral and equiangular.

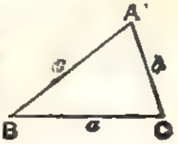
7. Triangles are thus classified with regard to their sides: A triangle is said to be
   equilateral, when *all* its sides are equal;
   isosceles, when *two* of its sides are equal;
   scalene, when its sides are all unequal.

Equilateral Triangle.

Isosceles Triangle.

Scalene Triangle.

In a triangle ABC, the letters A, B, C often denote the *magnitude* of the several angles (as measured in degrees); and the letters a, b, c the *lengths* of the opposite sides (as measured in inches, centimetres, or some other unit of length).



Any one of the angular points of a triangle may be regarded as its **vertex**; and the opposite side is then called the base.

In an *isosceles* triangle the term *vertex* is usually applied to the point at which the equal sides intersect; and the vertical angle is the angle included by them.

8. Triangles are thus classified with regard to their angles:

A triangle is said to be

right-angled, when one of its angles is a right angle;
obtuse-angled, when one of its angles is obtuse;
acute-angled, when *all three* of its angles are acute.

[It will be seen hereafter (Theorem 8. Cor. 1) *that every triangle must have at least two acute angles.*]

Right-angled Triangle. Obtuse-angled Triangle. Acute-angled Triangle.

In a right-angled triangle the side opposite to the right angle is called the hypotenuse.

9. In any triangle the straight line joining a vertex to the middle point of the opposite side is called a **median.**

THE COMPARISON OF TWO TRIANGLES.

(i) The three sides and three angles of a triangle are called its six parts. A triangle may also be considered with regard to its area.

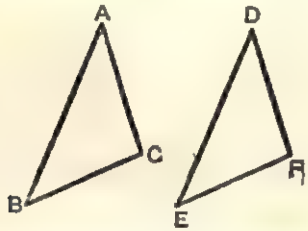
(ii) Two triangles are said to be equal in all respects, when one may be so placed upon the other as to exactly coincide with it; in which case each part of the first triangle is equal to the corresponding part (namely that with which it coincides) of the other; and the triangles are equal in area.

In two such triangles corresponding sides are *opposite to equal angles,* and corresponding angles are *opposite to equal sides.*

Triangles which may thus be made to coincide by superposition are said to be **identically equal** or **congruent.**

Theorem 4. [Euclid I. 4.]

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles included by those sides equal, then the triangles are equal in all respects.



Let ABC, DEF be two triangles in which

AB = DE,
AC = DF,

and the included angle BAC = the included angle EDF.

It is required to prove that the $\triangle$ ABC = *the* $\triangle$ DEF *in all respects.*

Proof. Apply the $\triangle$ ABC to the $\triangle$ DEF, so that the point A falls on the point D, and the side AB along the side DE.

Then because AB = DE,
$\therefore$ the point B must coincide with the point E.

And because AB falls along DE,
and the $\angle$ BAC = the $\angle$ EDF,
$\therefore$ AC must fall along DF.

And because AC = DF,
$\therefore$ the point C must coincide with the point F.

Then since B coincides with E, and C with F,
$\therefore$ the side BC must coincide with the side EF.

Hence the $\triangle$ ABC coincides with the $\triangle$ DEF, and is therefore equal to it in all respects.

Q.E.D.

Obs. In this Theorem we must carefully observe what is *given* and what is *proved*.

$$\textit{Given that} \begin{cases} AB = DE, \\ AC = DF, \\ \text{and the } \angle BAC = \text{the } \angle EDF. \end{cases}$$

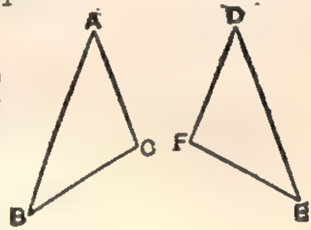
From these data we prove that the triangles coincide on superposition.

$$\text{Hence we } \textit{conclude} \text{ that} \begin{cases} BC = EF, \\ \text{the } \angle ABC = \text{the } \angle DEF, \\ \text{and the } \angle ACB = \text{the } \angle DFE; \end{cases}$$

also that the triangles are equal in area.

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal.

NOTE. The adjoining diagram shews that in order to make two congruent triangles coincide, it may be necessary to *reverse*, that is, turn over one of them before superposition.



EXERCISES.

1. *Shew that the bisector of the vertical angle of an isosceles triangle* (i) *bisects the base,* (ii) *is perpendicular to the base.*

2. *Let* O *be the middle point of a straight line* AB, *and let* OC *be perpendicular to it. Then if* P *is any point in* OC, *prove that* PA = PB.

3. Assuming that the four sides of a square are equal, and that its angles are all right angles, prove that in the square ABCD, the diagonals AC, BD are equal.

4. ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD: prove that

(i) LM = MN. (ii) AM = DM.

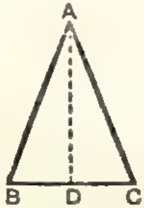
(iii) AN = AM. (iv) BN = DM.

[Draw a separate figure in each case.]

5. ABC is an isosceles triangle: from the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that BY = CX.

THEOREM 5. [Euclid I. 5.]

The angles at the base of an isosceles triangle are equal.



Let ABC be an isosceles triangle, in which the side AB = the side AC.

It is required to prove that the ∠ ABC = *the* ∠ ACB.

Suppose that AD is the line which bisects the ∠ BAC, and let it meet BC in D.

1st Proof. Then in the △^s BAD, CAD,

because { BA = CA,
AD is common to both triangles,
and the included ∠ BAD = the included ∠ CAD;

∴ the triangles are equal in all respects; *Theor.* 4.
so that the ∠ ABD = the ∠ ACD.

Q.E.D.

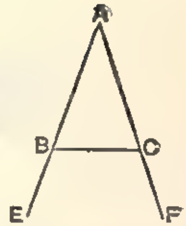
2nd Proof. Suppose the △ ABC to be folded about AD.

Then since the ∠ BAD = the ∠ CAD,
∴ AB must fall along AC.

And since AB = AC,
∴ B must fall on C, and consequently DB on DC.

∴ the ∠ ABD will coincide with the ∠ ACD, and is therefore equal to it.

Q.E.D.



COROLLARY 1. *If the equal sides* AB, AC *of an isosceles triangle are produced, the exterior angles* EBC, FCB *are equal; for they are the supplements of the equal angles at the base.*

COROLLARY 2. *If a triangle is equilateral, it is also equiangular.*

DEFINITION. A figure is said to be **symmetrical about a line** when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence.

The straight line is called an **axis of symmetry**.

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis.

Theorem 5 proves that *an isosceles triangle is symmetrical about the bisector of its* VERTICAL *angle.*

An equilateral triangle is symmetrical about the bisector of ANY ONE *of its angles.*

EXERCISES.

1. ABCD is a four-sided figure whose sides are all equal, and the diagonal BD is drawn: shew that

(i) the angle ABD = the angle ADB;
(ii) the angle CBD = the angle CDB;
(iii) the angle ABC = the angle ADC.

2. ABC, DBC are two isosceles triangles drawn on the same base BC, but on *opposite* sides of it: prove (by means of Theorem 5) that
the angle ABD = the angle ACD.

3. ABC, DBC are two isosceles triangles drawn on the same base BC and on *the same* side of it: employ Theorem 5 to prove that
the angle ABD = the angle ACD.

4. AB, AC are the equal sides of an isosceles triangle ABC; and L, M, N are the middle points of AB, BC, and CA respectively: prove that

(i) LM = NM. (ii) BN = CL.
(iii) the angle ALM = the angle ANM.

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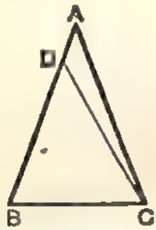
THEOREM 6. [Euclid I. 6.]

If two angles of a triangle are equal to one another, then the sides which are opposite to the equal angles are equal to one another.



Let ABC be a triangle in which

the $\angle$ ABC = the $\angle$ ACB.

It is required to prove that the side AC = *the side* AB.

If AC and AB are not equal, suppose that AB is the greater.

From BA cut off BD equal to AC.

Join DC.

Proof. Then in the $\triangle^s$ DBC, ACB,

because $\begin{cases} \text{DB} = \text{AC}, \\ \text{BC is common to both,} \\ \text{and the included } \angle \text{ DBC} = \text{the included } \angle \text{ ACB;} \end{cases}$

$\therefore$ the $\triangle$ DBC = the $\triangle$ ACB in area, *Theor.* 4.

the part equal to the whole; which is absurd.

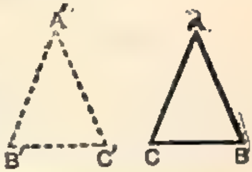
$\therefore$ AB is not unequal to AC;

that is, AB = AC.

Q.E.D.

COROLLARY. *Hence if a triangle is equiangular it is also equilateral.*

NOTE ON THEOREMS 5 AND 6.



Theorems 5 and 6 may be verified experimentally by cutting out the given $\triangle ABC$, and, after turning it over, fitting it *thus reversed* into the vacant space left in the paper.

Suppose $A'B'C'$ to be the original position of the $\triangle ABC$, and let ACB represent the triangle when reversed.

In Theorem 5, it will be found on applying A to A′ that C may be made to fall on B′, and B on C′.

In Theorem 6, on applying C to B′ and B to C′ we find that A will fall on A′.

In either case the given triangle *reversed* will coincide with its own "trace," so that the side and angle on the *left* are respectively equal to the side and angle on the *right*.

NOTE ON A THEOREM AND ITS CONVERSE.

The enunciation of a theorem consists of two clauses. The first clause tells us what we are to *assume*, and is called the **hypothesis**; the second tells us what *it is required to prove*, and is called the conclusion.

For example, the enunciation of Theorem 5 assumes that in a certain triangle ABC *the side* $AB =$ *the side* AC: this is the *hypothesis*. From this it is required to prove that *the angle* $ABC =$ *the angle* ACB: this is the *conclusion*.

If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the **converse** of the first.

For example, in Theorem 5

it is *assumed* that $AB = AC$;
it is *required to prove* that the angle $ABC =$ the angle ACB.

Now in Theorem 6

it is *assumed* that the angle $ABC =$ the angle ACB;
it is *required to prove* that $AB = AC$.

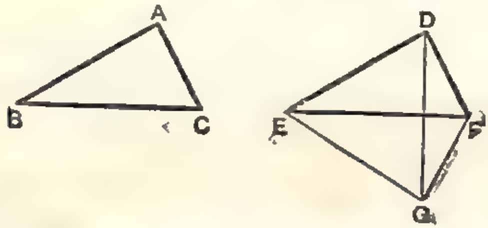
Thus we see that Theorem 6 is the converse of Theorem 5; for the *hypothesis of each is the conclusion of the other.*

In Theorem 6 we employ an *indirect method of proof* frequently used in geometry. It consists in shewing that the theorem *cannot be untrue*; since, if it were, we should be led to some *impossible conclusion.* This form of proof is known as **Reductio ad Absurdum**, and is most commonly used in demonstrating the converse of some foregoing theorem.

It must not however be supposed that if a theorem is true, its converse is *necessarily* true. [See p. 25.]

THEOREM 7. [Euclid I. 8.]

If two triangles have the three sides of the one equal to the three sides of the other, each to each, they are equal in all respects.



Let ABC, DEF be two triangles in which

AB = DE,
AC = DF,
BC = EF.

It is required to prove that the triangles are equal in all respects.

Proof. Apply the $\triangle$ ABC to the $\triangle$ DEF,
so that B falls on E, and BC along EF, and
so that A is on the side of EF opposite to D.
Then because BC = EF, C must fall on F.

Let GEF be the new position of the $\triangle$ ABC.
Join DG.

Because ED = EG,
$\therefore$ the $\angle$ EDG = the $\angle$ EGD. *Theor.* 5.

Again, because FD = FG,
$\therefore$ the $\angle$ FDG = the $\angle$ FGD.

Hence the whole $\angle$ EDF = the whole $\angle$ EGF,
that is, the $\angle$ EDF = the $\angle$ BAC.

Then in the $\triangle^s$ BAC, EDF;

because { BA = ED,
AC = DF,
and the included $\angle$ BAC = the included $\angle$ EDF;

$\therefore$ the triangles are equal in all respects. *Theor.* 4.

Q.E.D.

Obs. In this Theorem

it is *given* that AB = DE, BC = EF, CA = FD;

and we *prove* that ∠C = ∠F, ∠A = ∠D, ∠B = ∠E.

Also the triangles are equal in area.

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal.

Note 1. We have taken the case in which DG falls within the ∠ˢ EDF, EGF.

Two other cases might arise:

(i) DG might fall outside the ∠ˢ EDF, EGF [as in Fig. 1].

(ii) DG might coincide with DF, FG [as in Fig. 2].

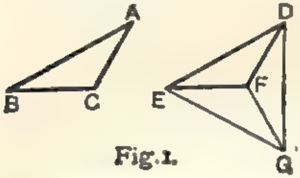


Fig. 1.

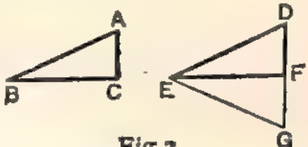


Fig. 2.

These cases will arise only when the given triangles are obtuse angled or right-angled; and (as will be seen hereafter) not even then, if we begin by choosing for superposition the *greatest* side of the △ ABC, as in the diagram of page 24.

Note 2. Two triangles are said to be equiangular to one another when the angles of one are respectively equal to the angles of the other.

Hence if two triangles have the three sides of one severally equal to the three sides of the other, the triangles are equiangular to one another.

The student should state the converse theorem, and shew by a diagram that the converse is not necessarily true.

*** *At this stage Problems* 1–5 *and* 8 [*see page* 70] *may conveniently be taken, the proofs affording good illustrations of the Identical Equality of Two Triangles.*

EXERCISES.

On The Identical Equality of Two Triangles. Theorems 4 and 7.

(*Theoretical.*)

1. Shew that the straight line which joins the vertex of an isosceles triangle to the middle point of the base,
   (i) bisects the vertical angle:
   (ii) is perpendicular to the base.

2. If ABCD is a rhombus, that is, an equilateral foursided figure; shew, by drawing the diagonal AC, that
   (i) the angle ABC = the angle ADC;
   (ii) AC bisects each of the angles BAD, BCD.

3. If in a quadrilateral ABCD the opposite sides are equal, namely AB = CD and AD = CB; prove that the angle ADC = the angle ABC.

4. If ABC and DBC are two isosceles triangles drawn on the same base BC, prove (by means of Theorem 7) that the angle ABD = the angle ACD, taking (i) the case where the triangles are on the *same* side of BC, (ii) the case where they are on *opposite* sides of BC.

5. If ABC, DBC are two isosceles triangles drawn on opposite sides of the same base BC, and if AD be joined, prove that each of the angles BAC, BDC will be divided into two equal parts.

6. Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides, are equal to one another.

7. Two given points in the base of an isosceles triangle are equidistant from the extremities of the base: shew that they are also equidistant from the vertex.

8. Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

9. ABC is an isosceles triangle having AB equal to AC; and the angles at B and C are bisected by BO and CO: shew that
   (i) BO = CO;
   (ii) AO bisects the angle BAC.

10. Shew that the diagonals of a rhombus [see Ex. 2] bisect one another at right angles.

11. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF; and FB, EC are joined: shew that FB is equal to EC.

EXERCISES ON TRIANGLES.

(Numerical and Graphical.)

1. Draw a triangle ABC, having given $a=2{\cdot}0''$, $b=2{\cdot}1''$, $c=1{\cdot}3''$. Measure the angles, and find their sum.

2. In the triangle ABC, $a=7{\cdot}5$ cm., $b=7{\cdot}0$ cm., and $c=6{\cdot}5$ cm. Draw and measure the perpendicular from B on CA.

3. Draw a triangle ABC, in which $a=7$ cm., $b=6$ cm., $C=65°$.

How would you prove theoretically that any two triangles having these parts are alike in size and shape? Invent some experimental illustration.

4. Draw a triangle from the following data: $b=2''$, $c=2{\cdot}5''$, $A=57°$; and measure a, B, and C.

Draw a second triangle, using as data the values just found for a, B, and C; and measure b, c, and A. What conclusion do you draw?

5. A ladder, whose foot is placed 12 feet from the base of a house, reaches to a window 35 feet above the ground. Draw a plan in which 1″ represents 10 ft.; and find by measurement the length of the ladder.

6. I go due North 99 metres, then due East 20 metres. Plot my course (scale 1 cm. to 10 metres), and find by measurement as nearly as you can how far I am from my starting point.

7. When the sun is 42° above the horizon, a vertical pole casts a shadow 30 ft. long. Represent this on a diagram (scale 1″ to 10 ft.); and find by measurement the approximate height of the pole.

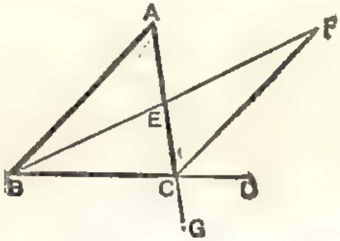
8. From a point A a surveyor goes 150 yards due East to B; then 300 yards due North to C; finally 450 yards due West to D. Plot h course (scale 1″ to 100 yards); and find roughly how far D is from A. Measure the angle DAB, and say in what direction D bears from A.

9. B and C are two points, known to be 260 yards apart, on a straight shore. A is a vessel at anchor. The angles CBA, BCA are observed to be 33° and 81° respectively. Find graphically the approximate distance of the vessel from the points B and C, and from the nearest point on shore.

10. In surveying a park it is required to find the distance between two points A and B; but as a lake intervenes, a direct measurement cannot be made. The surveyor therefore takes a third point C, from which both A and B are accessible, and he finds CA=245 yards, CB=320 yards, and the angle ACB=42°. Ascertain from a plan the approximate distance between A and B.

THEOREM 8. [Euclid I. 16.]

If one side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles.



Let ABC be a triangle, and let BC be produced to D.

It is required to prove that the exterior ∠ ACD *is greater than either of the interior opposite* ∠ˢ ABC, BAC.

Suppose E to be the middle point of AC.
Join BE; and produce it to F, making EF equal to BE.
Join FC.

Proof. Then in the △ˢ AEB, CEF,

because $\begin{cases} AE = CE, \\ EB = EF, \\ \text{and the } \angle AEB = \text{the vertically opposite } \angle CEF; \end{cases}$

∴ the triangles are equal in all respects; *Theor.* 4.
so that the ∠ BAE = the ∠ ECF.

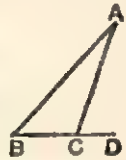
But the ∠ ECD is greater than the ∠ ECF;
∴ the ∠ ECD is greater than the ∠ BAE;
that is, the ∠ ACD is greater than the ∠ BAC.

In the same way, if AC is produced to G, by supposing A to be joined to the middle point of BC, it may be proved that the ∠ BCG is greater than the ∠ ABC.

But the ∠ BCG = the vertically opposite ∠ ACD.
∴ the ∠ ACD is greater than the ∠ ABC.

Q.E.D.

COROLLARY 1. *Any two angles of a triangle are together less than two right angles.*

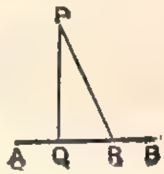


For the $\angle$ABC is less than the $\angle$ACD: *Proved.*
to each add the $\angle$ACB.
Then the $\angle$s ABC, ACB are less than the $\angle$s ACD, ACB,
therefore, less than two right angles.

COROLLARY 2. *Every triangle must have at least two acute angles.*

For if one angle is obtuse or a right angle, then by Cor. 1 each of the other angles must be less than a right angle.

COROLLARY 3. *Only one perpendicular can be drawn to a straight line from a given point outside it.*



If two perpendiculars could be drawn to AB from P, we should have a triangle PQR in which each of the $\angle$s PQR, PRQ would be a right angle, which is impossible.

EXERCISES.

1. Prove Corollary 1 by joining the vertex A to any point in the base BC.

2. ABC *is a triangle and* D *any point within it. If* BD *and* CD *are joined, the angle* BDC *is greater than the angle* BAC. Prove this

   (i) by producing BD to meet AC.

   (ii) by joining AD, and producing it towards the base.

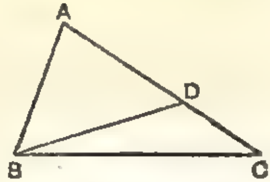
3. If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles.

4. To a given straight line there cannot be drawn from a point outside it more than two straight lines of the same given length.

5. If the equal sides of an isosceles triangle are produced, the exterior angles must be obtuse.

THEOREM 9. [Euclid I. 18.]

If one side of a triangle is greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less



Let ABC be a triangle, in which the side AC is greater than the side AB.

It is required to prove that the ∠ ABC *is greater than the* ∠ ACB.

From AC cut off AD equal to AB.
Join BD.

Proof. Because AB = AD,

∴ the ∠ ABD = the ∠ ADB. *Theor.* 5.

But the exterior ∠ ADB of the △ BDC is greater than the interior opposite ∠ DCB, that is, greater than the ∠ ACB.

∴ the ∠ ABD is greater than the ∠ ACB.

Still more then is the ∠ ABC greater than the ∠ ACB.

Q.E.D.

Obs. The mode of demonstration used in the following Theorem is known as the **Proof by Exhaustion**. It is applicable to cases in which *one* of certain suppositions must necessarily be true; and it consists in shewing that each of these suppositions is false *with one exception*: hence the truth of the remaining supposition is inferred.

THEOREM 10. [Euclid I. 19.]

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less.

Let ABC be a triangle, in which the ∠ABC is greater than the ∠ACB.

It is required to prove that the side AC *is greater than the side* AB.

Proof. If AC is not greater than AB,
it must be either equal to, or less than AB.

Now if AC were equal to AB,
then the ∠ABC would be equal to the ∠ACB; *Theor.* 5.
but, by hypothesis, it is not.

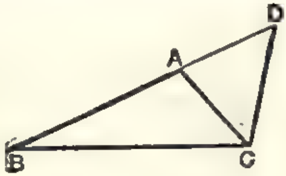
Again, if AC were less than AB,
then the ∠ABC would be less than the ∠ACB; *Theor.* 9.
but, by hypothesis, it is not.

That is, AC is neither equal to, nor less than AB.
∴ AC is greater than AB. Q.E.D.

[For Exercises on Theorems 9 and 10 see page 34.]

THEOREM 11. [Euclid I. 20.]

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

It is required to prove that any two of its sides are together greater than the third side.

It is enough to shew that if BC is the greatest side, then BA, AC are together greater than BC.

Produce BA to D, making AD equal to AC.
Join DC.

Proof. Because AD = AC,
∴ the ∠ACD = the ∠ADC. *Theor.* 5.

But the ∠BCD is greater than the ∠ACD;
∴ the ∠BCD is greater than the ∠ADC,
that is, than the ∠BDC.

Hence from the △BDC,
BD is greater than BC. *Theor.* 10.

But BD = BA and AC together;
∴ BA and AC are together greater than BC.

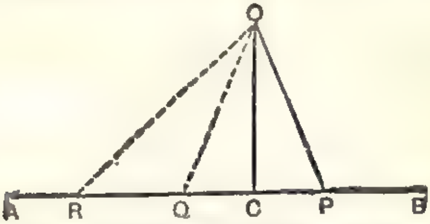
Q.E.D.

NOTE. This proof may serve as an exercise, but the truth of the Theorem is really self-evident. For to go from B to C along the straight line BC is clearly shorter than to go from B to A and then from A to C. In other words

The shortest distance between two points is the straight line which joins them.

THEOREM 12.

Of all straight lines drawn from a given point to a given straight line the perpendicular is the least.



Let OC be the perpendicular, and OP *any* oblique, drawn from the given point O to the given straight line AB.

It is required to prove that OC *is less than* OP.

Proof. In the $\triangle$ OCP, since the $\angle$ OCP is a right angle,

$\therefore$ the $\angle$ OPC is less than a right angle; *Theor.* 8. *Cor.*

that is, the $\angle$ OPC is less than the $\angle$ OCP.

$\therefore$ OC is less than OP. *Theor.* 10.

Q.E.D.

COROLLARY 1. Hence conversely, since there can be only one perpendicular and one shortest line from O to AB,

If OC *is the shortest straight line from* O *to* AB, *then* OC *is perpendicular to* AB.

COROLLARY 2. *Two obliques* OP, OQ, *which cut* AB *at equal distances from* C *the foot of the perpendicular, are equal.*

The $\triangle$s OCP, OCQ may be shewn to be congruent by Theorem 4;

hence OP=OQ.

COROLLARY 3. *Of two obliques* OQ, OR, *if* OR *cuts* AB *at the greater distance from* C *the foot of the perpendicular, then* OR *is greater than* OQ.

The $\angle$OQC is acute, $\therefore$ the $\angle$OQR is obtuse;

$\therefore$ the $\angle$OQR is greater than the $\angle$ORQ;

$\therefore$ OR is greater than OQ.

EXERCISES ON INEQUALITIES IN A TRIANGLE.

1. *The hypotenuse is the greatest side of a right-angled triangle.*

2. *The greatest side of any triangle makes acute angles with each of the other sides.*

3. *If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, then these straight lines are together less than the other two sides of the triangle.*

4. BC, the base of an isosceles triangle ABC, is produced to any point D; shew that AD is greater than either of the equal sides.

5. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

6. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

7. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: shew that, if AB is greater than AC, then OB is greater than OC.

8. The difference of any two sides of a triangle is less than the third side.

9. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

10. The perimeter of a quadrilateral is greater than the sum of its diagonals.

11. ABC is a triangle, and the vertical angle BAC is bisected by a line which meets BC in X; shew that BA is greater than BX, and CA greater than CX. Hence obtain a proof of Theorem 11.

12. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

13. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point. Prove this, and point out the exceptional case.

14. *In a triangle any two sides are together greater than twice the median which bisects the remaining side.*

[Produce the median, and complete the construction after the manner of Theorem 8.]

15. *In any triangle the sum of the medians is less than the perimeter.*

PARALLELS.

DEFINITION. Parallel straight lines are such as, being in the same plane, do not meet however far they are produced beyond both ends.

NOTE. Parallel lines must be *in the same plane*. For instance, two straight lines, one of which is drawn on a table and the other on the floor, would never meet if produced; but they are not for that reason necessarily parallel.

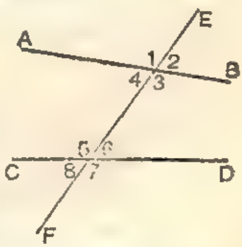
AXIOM. *Two intersecting straight lines cannot both be parallel to a third straight line.*

In other words:

Through a given point there can be only one straight line parallel to a given straight line.

This assumption is known as *Playfair's Axiom.*

DEFINITION. When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.



Thus in the adjoining figure,
1, 2, 7, 8 are called exterior angles,
3, 4, 5, 6 are called interior angles,
4 and 6 are said to be alternate angles; so also the angles 3 and 5 are alternate to one another.

Of the angles 2 and 6, 2 is referred to as the exterior angle, and 6 as the interior opposite angle on the same side of EF. Such angles are also known as corresponding angles. Similarly 7 and 3, 8 and 4, 1 and 5 are pairs of corresponding angles.

THEOREM 13. [Euclid I. 27 and 28.]

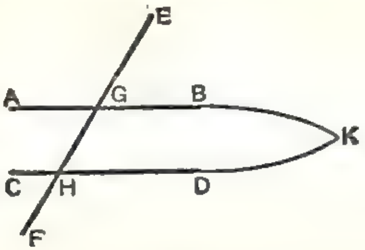
If a straight line cuts two other straight lines so as to make

(i) *the alternate angles equal,*

or (ii) *an exterior angle equal to the interior opposite angle on the same side of the cutting line,*

or (iii) *the interior angles on the same side equal to two right angles;*

then in each case the two straight lines are parallel.



(i) Let the straight line EGHF cut the two straight lines AB, CD at G and H so as to make the alternate $\angle^s$ AGH, GHD equal to one another.

It is required to prove that AB *and* CD *are parallel.*

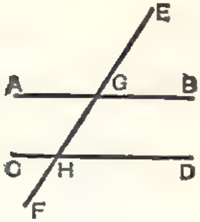
Proof. If AB and CD are not parallel, they will meet, if produced, either towards B and D, or towards A and C.

If possible, let AB and CD, when produced, meet towards B and D, at the point K.

Then KGH is a triangle, of which one side KG is produced to A;
$\therefore$ the exterior $\angle$AGH is greater than the interior opposite $\angle$GHK; but, by hypothesis, it is not greater.

$\therefore$ AB and CD cannot meet when produced towards B and D.
Similarly it may be shewn that they cannot meet towards A and C:

$\therefore$ AB and CD are parallel.



(ii) Let the exterior ∠ EGB = the interior opposite ∠ GHD.

It is required to prove that AB *and* CD *are parallel.*

Proof. Because the ∠ EGB = the ∠ GHD,
and the ∠ EGB = the vertically opposite ∠ AGH;
∴ the ∠ AGH = the ∠ GHD:

and these are alternate angles;
∴ AB and CD are parallel.

(iii) Let the two interior ∠ˢ BGH, GHD be together equal to two right angles.

It is required to prove that AB *and* CD *are parallel.*

Proof. Because the ∠ˢ BGH, GHD together = two right angles;
and because the adjacent ∠ˢ BGH, AGH together = two right angles;
∴ the ∠ˢ BGH, AGH together = the ∠ˢ BGH, GHD.

From these equals take the ∠ BGH;
then the remaining ∠ AGH = the remaining ∠ GHD:

and these are alternate angles;
∴ AB and CD are parallel.

Q.E.D.

DEFINITION. A straight line drawn across a set of given lines is called a **transversal.**

For instance, in the above diagram the line EGHF, which crosses the given lines AB, CD is a transversal.

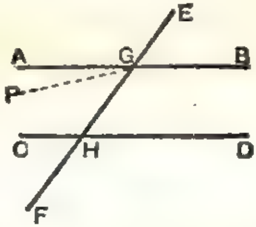
Theorem 14. [Euclid I. 29.]

If a straight line cuts two parallel lines, it makes

(i) *the alternate angles equal to one another;*

(ii) *the exterior angle equal to the interior opposite angle on the same side of the cutting line;*

(iii) *the two interior angles on the same side together equal to two right angles.*



Let the straight lines AB, CD be parallel, and let the straight line EGHF cut them.

It is required to prove that

(i) *the* $\angle$ AGH = *the alternate* $\angle$ GHD;

(ii) *the exterior* $\angle$ EGB = *the interior opposite* $\angle$ GHD;

(iii) *the two interior* $\angle^s$ BGH, GHD *together = two right angles.*

Proof. (i) If the $\angle$ AGH is not equal to the $\angle$ GHD, suppose the $\angle$ PGH equal to the $\angle$ GHD, and alternate to it;

then PG and CD are parallel. *Theor.* 13.

But, by hypothesis, AB and CD are parallel;

$\therefore$ the two intersecting straight lines AG, PG are both parallel to CD: which is impossible. *Playfair's Axiom.*

$\therefore$ the $\angle$ AGH is not unequal to the $\angle$ GHD;

that is, the alternate $\angle^s$ AGH, GHD are equal.

(ii) Again, because the $\angle$ EGB = the vertically opposite $\angle$ AGH;

and the $\angle$ AGH = the alternate $\angle$ GHD; *Proved.*

$\therefore$ the exterior $\angle$ EGB = the interior opposite $\angle$ GHD.

(iii) Lastly, the $\angle$ EGB = the $\angle$ GHD; *Proved.*

add to each the $\angle$ BGH;

then the $\angle^s$ EGB, BGH together = the angles BGH, GHD.

But the adjacent $\angle^s$ EGB, BGH together = two right angles;

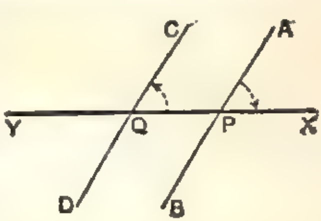
$\therefore$ the two interior $\angle^s$ BGH, GHD together = two right angles.

Q.E.D.

PARALLELS ILLUSTRATED BY ROTATION.

The direction of a straight line is determined by the angle which it makes with some given line of reference.

Thus the direction of AB, relatively to the given line YX, is given by the angle APX.



Now suppose that AB and CD in the adjoining diagram are parallel; then we have learned that the ext. $\angle$APX = the int. opp. $\angle$CQX; that is, AB and CD *make equal angles* with the line of reference YX.

This brings us to the leading idea connected with parallels:
Parallel straight lines have the same DIRECTION, *but differ in* POSITION.

The same idea may be illustrated thus:

Suppose AB to rotate about P through the $\angle$APX, so as to take the position XY. Thence let it rotate about Q *the opposite way* through the equal $\angle$XQC: it will now take the position CD. Thus AB may be brought into the position of CD by two rotations which, being equal and opposite, involve no final change of direction.

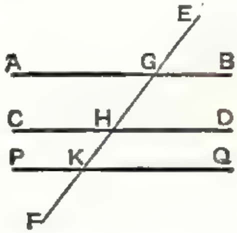
HYPOTHETICAL CONSTRUCTION. In the above diagram let AB be a fixed straight line, Q a fixed point, CD a straight line turning about Q, and YQPX any transversal through Q. Then as CD rotates, there must be one position in which the $\angle$ CQX = the fixed $\angle$ APX.

Hence through any given point we may assume a line to pass parallel to any given straight line.

Obs. If AB is a straight line, movements from A towards B, and from B towards A are said to be in opposite senses of the line AB.

THEOREM 15. [Euclid I. 30.]

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ.

It is required to prove that AB *and* CD *are parallel to one another.*

Draw a straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them,

$\therefore$ the $\angle$AGK = the alternate $\angle$GKQ.

And because CD and PQ are parallel, and EF meets them,

$\therefore$ the exterior $\angle$GHD = the interior opposite $\angle$GKQ.

$\therefore$ the $\angle$AGH = the $\angle$GHD ;

and these are alternate angles ;

$\therefore$ AB and CD are parallel.

Q.E.D.

NOTE. If PQ lies between AB and CD, the Proposition needs no proof ; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse.

For if AB and CD were not parallel, they would meet when produced. Then there would be two intersecting straight lines both parallel to a third straight line : which is impossible.

Therefore AB and CD never meet ; that is, they are parallel.

EXERCISES ON PARALLELS.

1. In the diagram of the previous page, if the angle EGB is 55°, express in degrees each of the angles GHC, HKQ, QKF.

2. *Straight lines which are perpendicular to the same straight line are parallel to one another.*

3. *If a straight line meet two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others.*

4. *Angles of which the arms are parallel, each to each, are either equal or supplementary.*

5. Two straight lines AB, CD bisect one another at O. Shew that the straight lines joining AC and BD are parallel.

6. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

7. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles.

8. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: shew the triangle AYZ is isosceles.

9. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles.

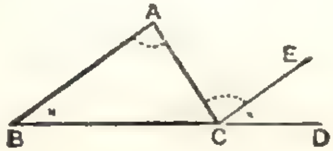
10. The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal; and the resulting figure is a rhombus.

11. AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected: if through any point X in DC a straight line YXZ is drawn parallel to AB and meeting the bisectors in Y and Z, shew that XY is equal to XZ.

12. Two straight rods PA, QB revolve about pivots at P and Q, PA making 12 complete revolutions a minute, and QB making 10. If they start parallel and pointing the same way, how long will it be before they are again parallel, (i) pointing opposite ways, (ii) pointing the same way?

THEOREM 16. [Euclid I. 32.]

The three angles of a triangle are together equal to two right angles.



Let ABC be a triangle.

It is required to prove that the three $\angle^s$ ABC, BCA, CAB *together* = *two right angles.*

Produce BC to any point D; and suppose CE to be the line through C parallel to BA.

Proof. Because BA and CE are parallel and AC meets them,
∴ the $\angle$ ACE = the alternate $\angle$ CAB.

Again, because BA and CE are parallel, and BD meets them,
∴ the exterior $\angle$ ECD = the interior opposite $\angle$ ABC.

∴ *the whole exterior* $\angle$ ACD = *the sum of the two interior opposite* $\angle^s$ CAB, ABC.

To each of these equals add the $\angle$ BCA;
then the $\angle^s$ BCA, ACD together = the three $\angle^s$ BCA, CAB, ABC.

But the adjacent $\angle^s$ BCA, ACD together = two right angles.
∴ the $\angle^s$ BCA, CAB, ABC together = two right angles.

Q.E.D.

Obs. In the course of this proof the following most important property has been established.

If a side of a triangle is produced the exterior angle is equal to the sum of the two interior opposite angles.

Namely, the ext. $\angle$ ACD = the $\angle$CAB + the $\angle$ ABC.

INFERENCES FROM THEOREM 16.

1. *If* A, B, *and* C *denote the number of degrees in the angles of a triangle,*

$$\textit{then } A + B + C = 180°.$$

2. *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other.*

3. *In any right-angled triangle the two acute angles are complementary.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

EXERCISES ON THEOREM 16.

1. Each angle of an equilateral triangle is two-thirds of a right angle, or 60°.

2. In a right-angled isosceles triangle each of the equal angles is 45°.

3. Two angles of a triangle are 36° and 123° respectively: deduce the third angle; and verify your result by measurement.

4. In a triangle ABC, the $\angle B = 111°$, the $\angle C = 42°$; deduce the $\angle A$, and verify by measurement.

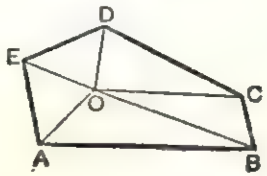
5. One side BC of a triangle ABC is produced to D. If the exterior angle ACD is 134°; and the angle BAC is 42°; find each of the remaining interior angles.

6. In the figure of Theorem 16, if the $\angle ACD = 118°$, and the $\angle B = 51°$, find the $\angle^s$ A and C; and check your results by measurement.

7. *Prove that the three angles of a triangle are together equal to two right angles by supposing a line drawn through the vertex parallel to the base.*

8. *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.*

COROLLARY 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Let ABCDE be a rectilineal figure of n sides.

It is required to prove that all the interior angles + 4 rt. $\angle^s$ $= 2n$ rt. $\angle^s$.

Take any point O within the figure, and join O to each of its vertices.

Then the figure is divided into n triangles.

And the three $\angle^s$ of each $\triangle$ together = 2 rt. $\angle^s$.
Hence *all* the $\angle^s$ of *all* the $\triangle^s$ together = $2n$ rt. $\angle^s$.

But all the $\angle^s$ of all the $\triangle^s$ make up all the interior angles of the figure together with the angles at O, which = 4 rt. $\angle^s$.

$\therefore$ all the int. $\angle^s$ of the figure + 4 rt. $\angle^s$ = $2n$ rt. $\angle^s$.

Q.E.D.

DEFINITION. A regular polygon is one which has all its sides equal and all its angles equal.

Thus if D denotes the number of degrees in each angle of a *regular* polygon of n sides, the above result may be stated thus:

$$n\text{D} + 360° = n \,.\, 180°.$$

EXAMPLE.

Find the number of degrees in each angle of

(i) a regular hexagon (6 sides);
(ii) a regular octagon (8 sides);
(iii) a regular decagon (10 sides).

EXERCISES ON THEOREM 16.

(Numerical and Graphical.)

1. ABC is a triangle in which the angles at B and C are respectively double and treble of the angle at A: find the number of degrees in each of these angles.

2. Express in degrees the angles of an isosceles triangle in which
(i) Each base angle is double of the vertical angle;
(ii) Each base angle is four times the vertical angle.

3. The base of a triangle is produced both ways, and the exterior angles are found to be 94° and 126°; deduce the vertical angle. Construct such a triangle, and check your result by measurement.

4. The sum of the angles at the base of a triangle is 162°, and their difference is 60°: find all the angles.

5. The angles at the base of a triangle are 84° and 62°; deduce (i) the vertical angle, (ii) the angle between the bisectors of the base angles. Check your results by construction and measurement.

6. In a triangle ABC, the angles at B and C are 74° and 62°; if AB and AC are produced, deduce the angle between the bisectors of the exterior angles. Check your result graphically.

7. Three angles of a quadrilateral are respectively $114\frac{1}{2}$°, 50°, and $75\frac{1}{2}$°; find the fourth angle.

8. In a quadrilateral ABCD, the angles at B, C, and D are respectively equal to 2A, 3A, and 4A; find all the angles.

9. Four angles of an irregular pentagon (5 sides) are 40°, 78°, 122°, and 135°; find the fifth angle.

10. In any *regular* polygon of n sides, each angle contains $\frac{2(n-2)}{n}$ right angles.
(i) Deduce this result from the Enunciation of Corollary 1.
(ii) Prove it independently by joining one vertex A to each of the others (except the two immediately adjacent to A), thus dividing the polygon into $n-2$ triangles.

11. How many sides have the regular polygons each of whose angles is (i) 108°, (ii) 156°?

12. Shew that the only *regular* figures which may be fitted together so as to form a plane surface are (i) *equilateral triangles*, (ii) *squares*, (iii) *regular hexagons*.

COROLLARY 2. *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*

1st Proof. Suppose, as before, that the figure has n sides; and consequently n vertices.

Now at *each* vertex

the interior ∠ + the exterior ∠ = 2 rt. ∠s;

and there are n vertices,

∴ the sum of the int. ∠s + the sum of the ext. ∠s = $2n$ rt. ∠s.

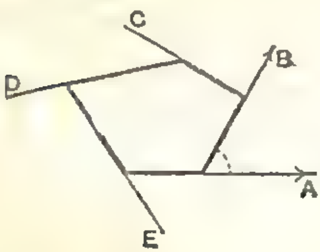
But by Corollary 1,

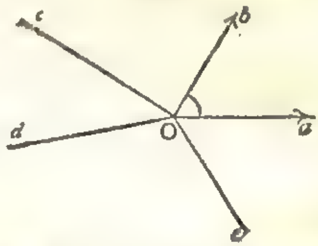
the sum of the int. ∠s + 4 rt. ∠s = $2n$ rt. ∠s;

∴ the sum of the ext. ∠s = 4 rt. ∠s.

Q.E.D.

2nd Proof.





Take any point O, and suppose Oa, Ob, Oc, Od, and Oe, are lines parallel to the sides marked, A, B, C, D, E (and drawn from O in the *sense* in which those sides were produced).

Then the exterior ∠ between the sides A and B = the ∠aOb.

And the other exterior ∠s = the ∠s bOc, cOd, dOe, eOa, respectively.

∴ the sum of the ext. ∠s = the sum of the ∠s at O

= 4 rt. ∠s.

EXERCISES.

1. If one side of a regular hexagon is produced, shew that the exterior angle is equal to the interior angle of an equilateral triangle.

2. Express in degrees the magnitude of each *exterior* angle of (i) a regular octagon, (ii) a regular decagon.

3. How many sides has a regular polygon if each exterior angle is (i) 30°, (ii) 24°?

4. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected, shew that the bisectors meet at right angles.

5. If the base of any triangle is produced both ways, shew that the sum of the two exterior angles *minus* the vertical angle is equal to two right angles.

6. In the triangle ABC the base angles at B and C are bisected by BO and CO respectively. Shew that the angle $BOC = 90° + \frac{A}{2}$.

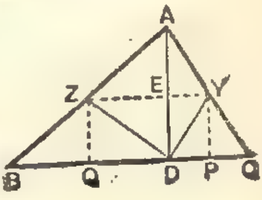
7. In the triangle ABC, the sides AB, AC are produced, and the exterior angles are bisected by BO and CO. Shew that the angle $BOC = 90° - \frac{A}{2}$.

8. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

9. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA; if DC is drawn, shew that BCD is a right angle.

10. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

EXPERIMENTAL PROOF OF THEOREM 16. [$A+B+C=180°$.]



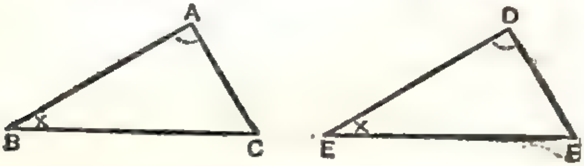
In the △ABC, AD is perp. to BC the greatest side. AD is bisected at right angles by ZY; and YP, ZQ are perpʳ. on BC.

If now the △ is folded about the three dotted lines, the ∠ˢ A, B, and C will coincide with the ∠ˢ ZDY, ZDQ, YDP;

∴ their sum is 180°.

THEOREM 17. [Euclid I. 26.]

If two triangles have two angles of one equal to two angles of the other, each to each, and any side of the first equal to the corresponding side of the other, the triangles are equal in all respects.



Let ABC, DEF be two triangles in which

the ∠ A = the ∠ D,
the ∠ B = the ∠ E,

also let the side BC = the corresponding side EF.

It is required to prove that the △ˢ ABC, DEF *are equal in all respects.*

Proof. The sum of the ∠ˢ A, B, and C
= 2 rt. ∠ˢ *Theor.* 16.
= the sum of the ∠ˢ D, E, and F;
and the ∠ˢ A and B = the ∠ˢ D and E respectively,
∴ the ∠ C = the ∠ F.

Apply the △ ABC to the △ DEF, so that B falls on E, and BC along EF.

Then because BC = EF,
∴ C must coincide with F.

And because the ∠ B = the ∠ E,
∴ BA must fall along ED.

And because the ∠ C = the ∠ F,
∴ CA must fall along FD.

∴ the point A, which falls both on ED and on FD, must coincide with D, the point in which these lines intersect.

∴ the △ ABC coincides with the △ DEF,
and is therefore equal to it in all respects.
So that AB = DE, and AC = DF;
and the △ ABC = the △ DEF in area. Q.E.D.

EXERCISES.

ON THE IDENTICAL EQUALITY OF TRIANGLES.

1. Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

2. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

3. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B: shew that AX is equal to BY.

4. If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

5. If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles.

6. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

[Produce the bisector, and complete the construction after the manner of Theorem 8.]

7. The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels.

8. A straight line drawn between two parallels and terminated by them, is bisected; shew that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point.

9. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

10. In a quadrilateral, ABCD, if AB = AD, and BC = DC: shew that the diagonal AC bisects each of the angles which it joins; and that AC is perpendicular to BD.

11. *A surveyor wishes to ascertain the breadth of a river which he cannot cross. Standing at a point* A *near the bank, he notes an object* B *immediately opposite on the other bank. He lays down a line* AC *of any length at right angles to* AB, *fixing a mark at* O *the middle point of* AC. *From* C *he walks along a line perpendicular to* AC *until he reaches a point* D *from which* O *and* B *are seen in the same direction. He now measures* CD: *prove that the result gives him the width of the river.*

ON THE IDENTICAL EQUALITY OF TRIANGLES.

Three cases of the congruence of triangles have been dealt with in Theorems 4, 7, 17, the results of which may be summarised as follows:

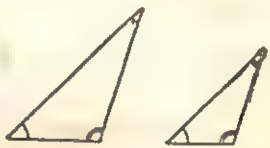
Two triangles are equal in all respects when the following three parts in each are severally equal:

1. *Two sides, and the included angle.* *Theorem* 4.
2. *The three sides.* *Theorem* 7.
3. *Two angles and one side, the side given in one triangle* CORRESPONDING *to that given in the other.* *Theorem* 17.

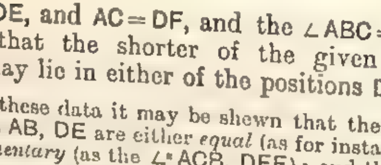
Two triangles are not, however, necessarily equal in all respects when *any three parts* of one are equal to the corresponding parts of the other.

For example:

(i) When the *three angles* of one are equal to the *three angles* of the other, each to each, the adjoining diagram shews that the triangles need not be equal in all respects.



(ii) When *two sides and one angle* in one are equal to *two sides and one angle* of the other, the given angles being *opposite* to equal sides, the diagram below shews that the triangles need not be equal in all respects.



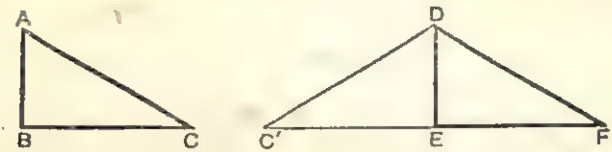
For if AB = DE, and AC = DF, and the ∠ABC = the ∠DEF, it will be seen that the shorter of the given sides in the triangle DEF may lie in either of the positions DF or DF′.

NOTE. From these data it may be shewn that the angles opposite to the equal sides AB, DE are either *equal* (as for instance the ∠ˢ ACB, DF′E) or *supplementary* (as the ∠ˢ ACB, DFE); and that in the former case the triangles are equal in all respects. This is called the ambiguous case in the congruence of triangles. [See Problem 9, p. 82.]

If the given angles at B and E are right angles, the ambiguity disappears. This exception is proved in the following Theorem.

THEOREM 18.

Two **right-angled** *triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects.*



Let ABC, DEF be two right-angled triangles, in which
the ∠ˢ ABC, DEF are right angles,
the hypotenuse AC = the hypotenuse DF,
and AB = DE.

It is required to prove that the △ˢ ABC, DEF *are equal in all respects.*

Proof. Apply the △ ABC to the △ DEF, so that AB falls on the equal line DE, and C on the side of DE opposite to F.
Let C′ be the point on which C falls.
Then DEC′ represents the △ ABC in its new position.

Since each of the ∠ˢ DEF, DEC′ is a right angle,
∴ EF and EC′ are in one straight line.

And in the △ C′DF, because DF = DC′ (*i.e.* AC),
∴ the ∠ DFC′ = the ∠ DC′F. *Theor.* 5.

Hence in the △ˢ DEF, DEC′,

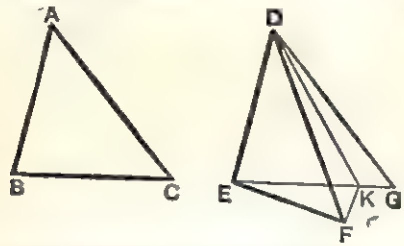
because { the ∠ DEF = the ∠ DEC′, being right angles;
the ∠ DFE = the ∠ DC′E, *Proved.*
and the side DE is common.

∴ the △ˢ DEF, DEC′ are equal in all respects; *Theor.* 17.
that is, the △ˢ DEF, ABC are equal in all respects.

Q.E.D.

*Theorem 19. [Euclid I. 24.]

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle included by the two sides of one greater than the angle included by the corresponding sides of the other; then the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two triangles, in which

BA = ED,

and AC = DF,

but the ∠ BAC is greater than the ∠ EDF.

It is required to prove that the base BC *is greater than the base* EF.

Proof. Apply the △ ABC to the △ DEF, so that A falls on D, and AB along DE.

Then because AB = DE, B must coincide with E.

Let DG, GE represent AC, CB in their new position.

Then if EG passes through F, EG is greater than EF;

that is, BC is greater than EF.

But if EG does not pass through F, suppose that DK bisects the ∠ FDG, and meets EG in K. Join FK.

Then in the △s FDK, GDK,

because { FD = GD,
DK is common to both,
and the included ∠ FDK = the included ∠ GDK;

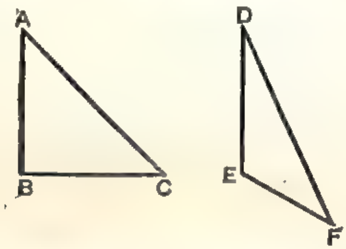
∴ FK = GK. *Theor.* 4.

Now the two sides EK, KF are greater than EF;

that is, EK, KG are greater than EF.

∴ EG (or BC) is greater than EF. Q.E.D.

Conversely, *if two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; then the angle contained by the sides of that which has the greater base, is greater than the angle contained by the corresponding sides of the other.*



Let ABC, DEF be two triangles in which
BA = ED,
and AC = DF,
but the base BC is greater than the base EF.

It is required to prove that the ∠ BAC *is greater than the* ∠ EDF.

Proof. If the ∠ BAC is not greater than the ∠ EDF,
it must be either equal to, or less than the ∠ EDF.

Now if the ∠ BAC were equal to the ∠ EDF,
then the base BC would be equal to the base EF; *Theor.* 4.
but, by hypothesis, it is not.

Again, if the ∠ BAC were less than the ∠ EDF,
then the base BC would be less than the base EF; *Theor.* 19.
but, by hypothesis, it is not.

That is, the ∠ BAC is neither equal to, nor less than the ∠ EDF;
∴ the ∠ BAC is greater than the ∠ EDF.

Q.E.D.

* *Theorems marked with an asterisk may be omitted or postponed at the discretion of the teacher.*

REVISION LESSON ON TRIANGLES.

1. State the properties of a triangle relating to

(i) the sum of its interior angles;

(ii) the sum of its exterior angles.

What property corresponds to (i) in a polygon of n sides? With what other figures does a triangle share the property (ii)?

2. Classify triangles with regard to their angles. Enunciate any Theorem or Corollary assumed in the classification.

3. Enunciate two Theorems in which from data relating to the sides a conclusion is drawn relating to the angles.

In the triangle ABC, if $a=3{\cdot}6$ cm., $b=2{\cdot}8$ cm., $c=3{\cdot}6$ cm., arrange the angles in order of their sizes (before measurement); and prove that the triangle is acute-angled.

4. Enunciate two Theorems in which from data relating to the angles a conclusion is drawn relating to the sides.

In the triangle ABC, if

(i) $A=48°$ and $B=51°$, find the third angle, and name the greatest side.

(ii) $A=B=62\frac{1}{2}°$, find the third angle, and arrange the sides in order of their lengths.

5. From which of the conditions given below may we conclude that the triangles ABC, A′B′C′ are identically equal? Point out where ambiguity arises; and draw the triangle ABC in each case.

(i) $\begin{cases} A=A'=71°. \\ B=B'=46°. \\ a=a'=3{\cdot}7 \text{ cm.} \end{cases}$ (ii) $\begin{cases} a=a'=4{\cdot}2 \text{ cm.} \\ b=b'=2{\cdot}4 \text{ cm.} \\ C=C'=81°. \end{cases}$ (iii) $\begin{cases} A=A'=36°. \\ B=B'=121°. \\ C=C'=23°. \end{cases}$

(iv) $\begin{cases} a=a'=3{\cdot}0 \text{ cm.} \\ b=b'=5{\cdot}2 \text{ cm.} \\ c=c'=4{\cdot}5 \text{ cm.} \end{cases}$ (v) $\begin{cases} B=B'=53°. \\ b=b'=4{\cdot}3 \text{ cm.} \\ c=c'=5{\cdot}0 \text{ cm.} \end{cases}$ (vi) $\begin{cases} C=C'=90°. \\ c=c'=5 \text{ cm.} \\ a=a'=3 \text{ cm.} \end{cases}$

6. Summarise the results of the last question by stating generally under what conditions two triangles

(i) are necessarily congruent;

(ii) may or may not be congruent.

7. *If two triangles have their angles equal, each to each, the triangles are not necessarily equal in all respects, because the three data are not* independent. Carefully explain this statement.

(*Miscellaneous Examples.*)

8. (i) *The perpendicular is the shortest line that can be drawn to a given straight line from a given point.*

(ii) **Obliques** *which make equal angles with the perpendicular are equal.*

(iii) *Of two obliques the less is that which makes the smaller angle with the perpendicular.*

9. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects.*

10. PQ is a perpendicular (4 cm. in length) to a straight line XY. Draw through P a series of obliques making with PQ the angles 15°, 30°, 45°, 60°, 75°. *Measure* the lengths of these obliques, and tabulate the results.

11. PAB is a triangle in which AB and AP have constant lengths 4 cm. and 3 cm. If AB is fixed, and AP rotates about A, trace the changes in PB, as the angle A increases from 0° to 180°.

Answer this question by drawing a series of figures, increasing A by increments of 30°. Measure PB in each case, and tabulate the results.

12. From B the foot of a flagstaff AB a horizontal line is drawn passing two points C and D which are 27 feet apart. The angles BCA and BDA are 65° and 40° respectively. Represent this on a diagram (scale 1 cm. to 10 ft.), and find by measurement the approximate height of the flagstaff.

13. From P, the top of a lighthouse PQ, two boats A and B are seen at anchor in a line due south of the lighthouse. It is known that PQ = 126 ft., $\angle$PAQ = 57°, $\angle$PBQ = 33°; hence draw a plan in which 1″ represents 100 ft., and find by measurement the distance between A and B to the nearest foot.

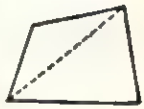
14. From a lighthouse L two ships A and B, which are 600 yards apart, are observed in directions S.W. and 15° East of South respectively. At the same time B is observed from A in a S.E. direction. Draw a plan (scale 1″ to 200 yds.), and find by measurement the distance of the lighthouse from each ship.

PARALLELOGRAMS.

DEFINITIONS.

1. A **quadrilateral** is a plane figure bounded by *four* straight lines.

The straight line which joins opposite angular points in a quadrilateral is called a **diagonal.**

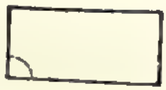


2. A **parallelogram** is a quadrilateral whose opposite sides are *parallel.*

[It will be proved hereafter that the opposite sides of a parallelogram are equal, and that its opposite angles are equal.]

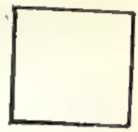
3. A **rectangle** is a parallelogram which has one of its angles a right angle.

[It will be proved hereafter that *all* the angles of a rectangle are right angles. See page 59.]



4. A **square** is a rectangle which has two adjacent sides equal.

[It will be proved that *all* the sides of a square are equal and all its angles right angles. See page 59.]

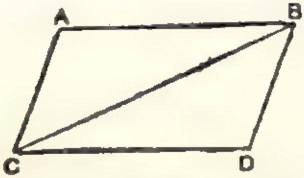


5. A **rhombus** is a quadrilateral which has all its sides equal, but its angles are not right angles.

6. A **trapezium** is a quadrilateral which has *one* pair of parallel sides.

THEOREM 20. [Euclid I. 33.]

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD.

It is required to prove that AC *and* BD *are equal and parallel.*

Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,

$\therefore$ the $\angle$ ABC = the alternate $\angle$ DCB.

Now in the $\triangle$s ABC, DCB,

because $\begin{cases} \text{AB} = \text{DC}, \\ \text{BC is common to both;} \\ \text{and the } \angle \text{ABC} = \text{the } \angle \text{DCB;} \end{cases}$ *Proved.*

$\therefore$ the triangles are equal in all respects;

so that AC = DB, ...(i)

and the $\angle$ ACB = $\angle$ DBC.

But these are alternate angles;

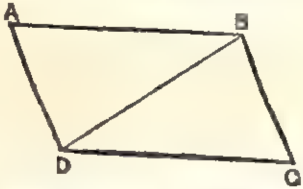
$\therefore$ AC and BD are parallel. ....(ii)

That is, AC and BD are both equal and parallel.

Q.E.D.

THEOREM 21. [Euclid I. 34.]

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ABCD be a parallelogram, of which BD is a diagonal.

It is required to prove that

(i) AB = CD, *and* AD = CB,
(ii) *the* ∠ BAD = *the* ∠ DCB,
(iii) *the* ∠ ADC = *the* ∠ CBA,
(iv) *the* △ ABD = *the* △ CDB *in area.*

Proof. Because AB and DC are parallel, and BD meets them,
∴ the ∠ ABD = the alternate ∠ CDB.

Again, because AD and BC are parallel, and BD meets them,
∴ the ∠ ADB = the alternate ∠ CBD.

Hence in the △^{s} ABD, CDB,

because $\left\{\begin{array}{l}\text{the } \angle \text{ABD} = \text{the } \angle \text{CDB,}\\ \text{the } \angle \text{ADB} = \text{the } \angle \text{CBD,}\\ \text{and BD is common to both;}\end{array}\right.$ *Proved.*

∴ the triangles are equal in all respects; *Theor.* 17.
so that AB = CD, and AD = CB; .... (i)
and the ∠ BAD = the ∠ DCB; .... (ii)
and the △ ABD = the △ CDB in area. ....(iv)

And because the ∠ ADB = the ∠ CBD,
and the ∠ CDB = the ∠ ABD, *Proved.*
∴ the whole ∠ ADC = the whole ∠ CBA ....(iii)

Q.E.D.

COROLLARY 1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

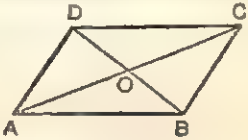
In other words:

All *the angles of a rectangle are right angles.*

For the sum of two consecutive $\angle^s = 2$ rt. $\angle^s$; (*Theor.* 14.)
∴, if one of these is a rt. angle, the other must be a rt. angle.
And the opposite angles of the par^m are equal;
∴ all the angles are right angles.

COROLLARY 2. *All the sides of a square are equal; and all its angles are right angles.*

COROLLARY 3. *The diagonals of a parallelogram bisect one another.*



Let the diagonals AC, BD of the par^m ABCD intersect at O.

To prove AO = OC, *and* BO = OD.

In the $\triangle^s$ AOB, COD,

because $\begin{cases} \text{the } \angle \text{OAB} = \text{the alt. } \angle \text{OCD}, \\ \text{the } \angle \text{AOB} = \text{vert. opp. } \angle \text{COD}, \\ \quad \text{and AB} = \text{the opp. side CD}; \end{cases}$

∴ OA = OC; and OB = OD. *Theor.* 17.

EXERCISES.

1. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

2. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

3. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

4. *The diagonals of a rhombus bisect one another at right angles.*

5. If the diagonals of a parallelogram are equal, all its angles are right angles.

6. In a parallelogram which is not rectangular the diagonals are unequal.

EXERCISES ON PARALLELS AND PARALLELOGRAMS.

(*Symmetry and Superposition.*)

1. Shew that by folding a rhombus about one of its diagonals the triangles on opposite sides of the crease may be made to coincide.

That is to say, prove that a rhombus is *symmetrical* about either diagonal.

2. Prove that the diagonals of a square are *axes of symmetry*. Name two other lines about which a square is symmetrical.

3. The diagonals of a rectangle divide the figure into two congruent triangles: is the diagonal, therefore, an axis of symmetry? About what two lines is a rectangle symmetrical?

4. Is there any axis about which an oblique parallelogram is symmetrical? Give reasons for your answer.

5. In a quadrilateral ABCD, AB=AD and CB=CD; but the sides are not all equal. Which of the diagonals (if either) is an axis of symmetry?

6. Prove by the method of superposition that

(i) *Two parallelograms are identically equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other.*

(ii) *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

7. Two quadrilaterals ABCD, EFGH have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH. Shew that the figures may be made to coincide with one another.

(*Miscellaneous Theoretical Examples.*)

8. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

9. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

10. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; shew that the figure AYCX is a parallelogram.

11. ABC and DEF are two triangles such that AB, BC are respectively equal to and parallel to DE, EF; shew that AC is equal and parallel to DF.

12. ABCD is a quadrilateral in which AB is parallel to DC, and AD equal but not parallel to BC; shew that

(i) the $\angle A +$ the $\angle C = 180° =$ the $\angle B +$ the $\angle D$;

(ii) the diagonal AC = the diagonal BD;

(iii) the quadrilateral is *symmetrical* about the straight line joining the middle points of AB and DC.

13. AP, BQ are straight rods of equal length, turning at equal rates (both clockwise) about two fixed pivots A and B respectively. If the rods start parallel but pointing in opposite senses, shew that

(i) they will always be parallel;

(ii) the line joining PQ will always pass through a certain fixed point.

(Miscellaneous Numerical and Graphical Examples.)

14. Calculate the angles of the triangle ABC, having given:

$$\text{int. } \angle A = \tfrac{8}{7} \text{ of ext. } \angle A; \quad 3B = 4C.$$

15. A yacht sailing due East changes her course successively by 63°, by 78°, by 119°, and by 64°, with a view to sailing round an island. What further change must be made to set her once more on an Easterly course?

16. If the sum of the interior angles of a rectilineal figure is equal to the sum of the exterior angles, how many sides has it, and why?

17. Draw, using your protractor, any five-sided figure ABCDE, in which

$$\angle B = 110°, \quad \angle C = 115°, \quad \angle D = 93°, \quad \angle E = 152°.$$

Verify by a construction with ruler and compasses that AE is parallel to BC, and account theoretically for this fact.

18. A and B are two fixed points, and two straight lines AP, BQ, unlimited towards P and Q, are pivoted at A and B. AP, starting from the direction AB, turns about A clockwise at the uniform rate of $7\frac{1}{2}°$ a second; and BQ, starting simultaneously from the direction BA, turns about B counter-clockwise at the rate of $3\frac{3}{4}°$ a second.

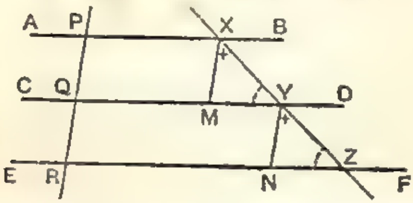
(i) How many seconds will elapse before AP and BQ are parallel?

(ii) Find graphically and by calculation the angle between AP and BQ twelve seconds from the start.

(iii) At what rate does this angle decrease?

THEOREM 22.

If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal.



Let the parallels AB, CD, EF cut off equal intercepts PQ, QR from the transversal PQR; and let XY, YZ be the corresponding intercepts cut off from any other transversal XYZ.

It is required to prove that XY = YZ.

Through X and Y let XM and YN be drawn parallel to PR.

Proof. Since CD and EF are parallel, and XZ meets them,

∴ the ∠XYM = the corresponding ∠YZN.

And since XM, YN are parallel, each being parallel to PR,

∴ the ∠MXY = the corresponding ∠NYZ.

Now the figures PM, QN are parallelograms,

∴ XM = the opp. side PQ, and YN = the opp. side QR;

and since by hypothesis PQ = QR,

∴ XM = YN.

Then in the △s XMY, YNZ,

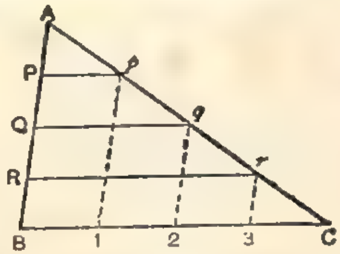
because $\begin{cases} \text{the } \angle XYM = \text{the } \angle YZN, \\ \text{the } \angle MXY = \text{the } \angle NYZ, \\ \text{and } XM = YN; \end{cases}$

∴ the triangles are identically equal; *Theor.* 17.

∴ XY = YZ.

Q.E.D.

COROLLARY. *In a triangle* ABC, *if a set of lines* P*p*, Q*q*, R*r*, ..., *drawn parallel to the base, divide one side* AB *into equal parts, they also divide the other side* AC *into equal parts.*



The lengths of the parallels P*p*, Q*q*, R*r*, ..., may thus be expressed in terms of the base BC.

Through *p*, *q*, and *r* let *p*1, *q*2, *r*3 be drawn par[l] to AB.

Then, by Theorem 22, these par[ls] divide BC into *four* equal parts, of which P*p* evidently contains *one*, Q*q* *two*, and R*r* *three*.

In other words,

$$Pp=\frac{1}{4}\cdot BC; \qquad Qq=\frac{2}{4}\cdot BC; \qquad Rr=\frac{3}{4}\cdot BC.$$

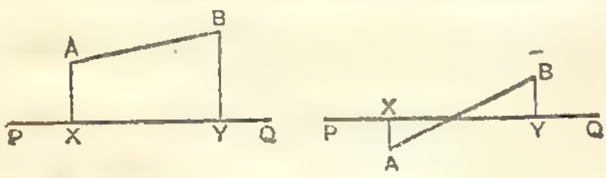
Similarly if the given par[ls] divide AB into *n* equal parts,

$$Pp=\frac{1}{n}\cdot BC, \qquad Qq=\frac{2}{n}\cdot BC, \qquad Rr=\frac{3}{n}\cdot BC; \text{ and so on.}$$

⁂ *Problem* 7, *p.* 78, *should now be worked.*

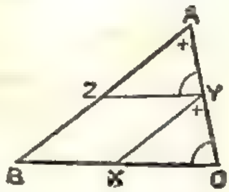
DEFINITION.

If from the extremities of a straight line AB perpendiculars AX, BY are drawn to a straight line PQ of indefinite length, then XY is said to be the orthogonal projection of AB on PQ.



EXERCISES ON PARALLELS AND PARALLELOGRAMS.

1. *The straight line drawn through the middle point of a side of a triangle, parallel to the base, bisects the remaining side.*

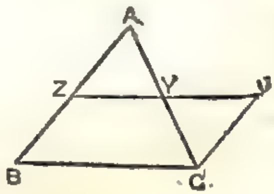


[This is an important particular case of Theorem 22.

In the $\triangle ABC$, if Z is the middle point of AB, and ZY is drawn par[l] to BC, we have to prove that $AY=YC$.

Draw YX par[l] to AB, and then prove the $\triangle^s$ ZAY, XYC congruent.]

2. *The straight line which joins the middle points of two sides of a triangle is parallel to the third side.*



[In the $\triangle ABC$, if Z, Y are the middle points of AB, AC, we have to prove ZY par[l] to BC.

Produce ZY to V, making YV equal to ZY, and join CV. Prove the $\triangle^s$ AYZ, CYV congruent; the rest follows at once.]

3. *The straight line which joins the middle points of two sides of a triangle is equal to half the third side.*

4. *Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal.*

5. *Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.*

6. ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: shew that BX and DY trisect the diagonal AC.

7. *If the middle points of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram.*

8. Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

9. From two points A and B, and from O the mid-point between them, perpendiculars AP, BQ, OX are drawn to a straight line CD. If AP, BQ measure respectively 4·2 cm. and 5·8 cm., *deduce* the length of OX, and verify your result by measurement.

Shew that $OX = \frac{1}{2}(AP + BQ)$ or $\frac{1}{2}(AP \sim BQ)$, according as A and B are on the *same* side, or on *opposite* sides of CD.

10. When three parallels cut off equal intercepts from two transversals, shew that of the three parallel lengths between the two transversals the middle one is the Arithmetic Mean of the other two.

11. *The parallel sides of a trapezium are* a *centimetres and* b *centimetres in length. Prove that the line joining the middle points of the oblique sides is parallel to the parallel sides, and that its length is* $\frac{1}{2}(a+b)$ *centimetres.*

12. OX and OY are two straight lines, and along OX five points 1, 2, 3, 4, 5 are marked at equal distances. Through these points parallels are drawn in any direction to meet OY. Measure the lengths of these parallels: take their average, and compare it with the length of the *third* parallel. Prove *geometrically* that the 3rd parallel is the mean of all five.

State the corresponding theorem for any odd number $(2n+1)$ of parallels so drawn.

13. From the angular points of a parallelogram perpendiculars are drawn to any straight line which is outside the parallelogram: shew that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum of those drawn from the other pair.

[Draw the diagonals, and from their point of intersection suppose a perpendicular drawn to the given straight line.]

14. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is constant, that is, the same whatever point in the base is taken.]

How would this property be modified if the given point were taken in the base *produced*?

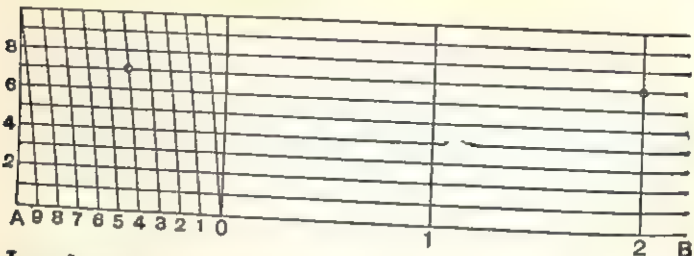
15. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant.

16. Equal and parallel lines have equal projections on any other straight line.

DIAGONAL SCALES.

Diagonal scales form an important application of Theorem 22. We shall illustrate their construction and use by describing a *Decimal Diagonal Scale to shew Inches, Tenths, and Hundredths.*

A straight line AB is divided (from A) into inches, and the points of division marked 0, 1, 2, ... The primary division 0A is subdivided into *tenths*, these secondary divisions being numbered (from 0) 1, 2, 3, ... 9. We may now read on AB *inches* and *tenths* of an inch.



In order to read *hundredths*, ten lines are taken at any equal intervals parallel to AB; and perpendiculars are drawn through 0, 1, 2, ...

The primary (or inch) division corresponding to 0A on the tenth parallel is now subdivided into *ten* equal parts; and diagonal lines are drawn, as in the diagram,
joining 0 to the *first* point of subdivision on the 10th parallel.
" 1 to the *second* " " " " ,
" 2 to the *third* " " " " ,
and so on. " " " " ;

The scale is now complete, and its use is shewn in the following example.

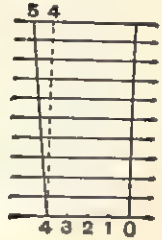
Example. To take from the scale a length of 2·47 *inches.*

(i) Place one point of the dividers at 2 in AB, and extend them till the other point reaches 4 in the subdivided inch 0A. We have now 2·4 inches in the dividers.

(ii) To get the remaining 7 *hundredths*, move the right-hand point up the perpendicular through 2 till it reaches the 7th parallel. Then extend the dividers till the left point reaches the diagonal 4 also on the 7th parallel. We have now 2·47 inches in the dividers.

REASON FOR THE ABOVE PROCESS.

The first step needs no explanation. The reason of the second is found in the Corollary of Theorem 22.



Joining the point 4 to the corresponding point on the tenth parallel, we have a triangle 4,4,5, of which one side 4,4 is divided into ten equal parts by a set of lines parallel to the base 4,5.

Therefore the lengths of the parallels between 4,4, and the diagonal 4,5 are $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots$ of the base, which is ·1 inch.

Hence these lengths are respectively

·01, ·02, ·03, ... of 1 inch.

Similarly, by means of this scale, the length of a given straight line may be measured to the nearest hundredth of an inch.

Again, if one inch-division on the scale is taken to represent 10 feet, then 2·47 inches on the scale will represent 24·7 feet. And if one inch-division on the scale represents 100 links, then 2·47 inches will represent 247 links. Thus a diagonal scale is of service in preparing plans of enclosures, buildings, or field-works, where it is necessary that every dimension of the actual object must be represented by a line of proportional length on the plan.

NOTE.

The subdivision of a diagonal scale need not be *decimal.*

For instance we might construct a diagonal scale to read centimetres, millimetres, and *quarters* of a millimetre; in which case we should take *four* parallels to the line AB.

[For Exercises on Linear Measurements see the following page.]

EXERCISES ON LINEAR MEASUREMENTS.

1. Draw straight lines whose lengths are 1·25 inches, 2·72 inches, 3·08 inches.

2. Draw a line 2·68 inches long, and measure its length in centimetres and the nearest millimetre.

3. Draw a line 5·7 cm. in length, and measure it in inches (to the nearest hundredth). Check your result by calculation, given that 1 cm. = 0·3937 inch.

4. Find by measurement the equivalent of 3·15 inches in centimetres and millimetres. Hence calculate (correct to two decimal places) the value of 1 cm. in inches.

5. Draw lines 2·9 cm. and 6·2 cm. in length, and measure them in inches. Use each equivalent to find the value of 1 inch in centimetres and millimetres, and take the average of your results.

6. A distance of 100 miles is represented on a map by 1 inch. Draw lines to represent distances of 336 miles and 408 miles.

7. If 1 inch on a map represents 1 kilometre, draw lines to represent 850 metres, 2980 metres, and 1010 metres.

8. A plan is drawn to the scale of 1 inch to 100 links. Measure in centimetres and millimetres a line representing 417 links.

9. Find to the nearest hundredth of an inch the length of a line which will represent 42·500 kilometres in a map drawn to the scale of 1 centimetre to 5 kilometres.

10. The distance from London to Oxford (in a direct line) is 55 miles. If this distance is represented on a map by 2·75 inches, to what scale is the map drawn? That is, how many miles will be represented by 1 inch? How many kilometres by 1 centimetre?
[1 cm. = 0·3937 inch; 1 km. = $\frac{5}{8}$ mile, nearly.]

11. On a map of France drawn to the scale 1 inch to 35 miles, the distance from Paris to Calais is represented by 4·2 inches. Find the distance accurately in miles, and approximately in kilometres, and express the scale in metric measure. [1 km. = $\frac{5}{8}$ mile, nearly.]

12. The distance from Exeter to Plymouth is $37\frac{1}{2}$ miles, and appears on a certain map to be $2\frac{1}{2}''$; and the distance from Lincoln to York is 88 km., and appears on another map to be 7 cm. Compare the scales of these maps in miles to the inch.

13. Draw a diagonal scale, 2 centimetres to represent 1 yard, shewing yards, feet, and inches.

PRACTICAL GEOMETRY.

PROBLEMS.

The following problems are to be solved with ruler and compasses only. No step requires the actual measurement of any line or angle; that is to say, the *constructions* are to be made without using either a graduated scale of length, or a protractor.

The problems are not merely to be studied as propositions; but the construction in every case is to be actually performed by the learner, great care being given to accuracy of drawing.

Each problem is followed by a *theoretical* proof; but the *results* of the work should always be verified by measurement, as a test of correct drawing. Accurate measurement is also required in applications of the problems.

In the diagrams of the problems lines which are inserted only for purposes of *proof* are dotted, to distinguish them from lines necessary to the construction.

For practical applications of the problems the student should be provided with the following instruments:

1. A flat ruler, one edge being graduated in centimetres and millimetres, and the other in inches and tenths.

2. Two set squares; one with angles of 45°, and the other with angles of 60° and 30°.

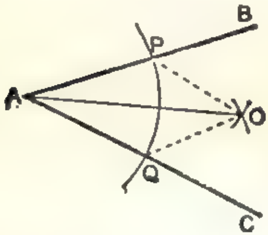
3. A pair of pencil compasses.

4. A pair of dividers, preferably with screw adjustment.

5. A semi-circular protractor.

PROBLEM 1.

To bisect a given angle.



Let BAC be the given angle to be bisected.

Construction. With centre A, and any radius, draw an arc of a circle cutting AB, AC at P and Q.

With centres P and Q, and radius PQ, draw two arcs cutting at O.

Join AO.

Then the ∠ BAC is bisected by AO.

Proof. Join PO, QO.

In the △s APO, AQO,

because { AP = AQ, being radii of a circle,
PO = QO, " " equal circles,
and AO is common ;

∴ the triangles are equal in all respects ; *Theor.* 7.

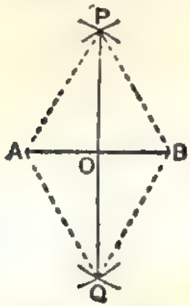
so that the ∠ PAO = the ∠ QAO ;

that is, the ∠ BAC is bisected by AO.

NOTE. PQ has been taken as the radius of the arcs drawn from the centres P and Q, and the intersection of these arcs determines the point O. *Any* radius, however, may be used instead of PQ, provided that it is great enough to secure the intersection of the arcs.

PROBLEM 2.

To bisect a given straight line



Let AB be the line to be bisected.

Construction. With centre A, and radius AB, draw two arcs, one on each side of AB.

With centre B, and radius BA, draw two arcs, one on each side of AB, cutting the first arcs at P and Q.

Join PQ, cutting AB at O.

Then AB is bisected at O.

Proof. Join AP, AQ, BP, BQ.

In the $\triangle^s$ APQ, BPQ,

because $\begin{cases} \text{AP} = \text{BP, being radii of equal circles,} \\ \text{AQ} = \text{BQ, for the same reason,} \\ \text{and PQ is common;} \end{cases}$

$\therefore$ the $\angle$ APQ = the $\angle$ BPQ. *Theor.* 7.

Again in the $\triangle^s$ APO, BPO,

because $\begin{cases} \text{AP} = \text{BP,} \\ \text{PO is common,} \\ \text{and the } \angle \text{APO} = \text{the } \angle \text{BPO;} \end{cases}$

$\therefore$ AO = OB; *Theor.* 4.

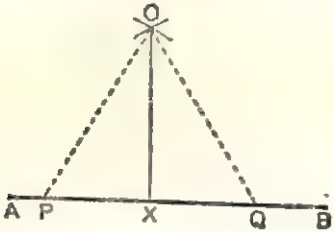
that is, AB is bisected at O.

NOTES. (i) AB was taken as the radius of the arcs drawn from the centres A and B, but *any* radius may be used provided that it is great enough to secure the intersection of the arcs which determine the points P and Q.

(ii) From the congruence of the $\triangle^s$ APO, BPO it follows that the $\angle$ AOP = the $\angle$ BOP. As these are adjacent angles, it follows that PQ *bisects* AB *at right angles.*

PROBLEM 3.

To draw a straight line perpendicular to a given straight line at a given point in it.



Let AB be the straight line, and X the point in it at which a perpendicular is to be drawn.

Construction. With centre X cut off from AB any two equal parts XP, XQ.

With centres P and Q, and radius PQ, draw two arcs cutting at O.

Join XO.
Then XO is perp. to AB.

Proof. Join OP, OQ.

In the $\triangle^s$ OXP, OXQ,

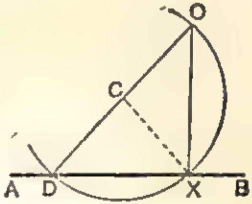
because $\begin{cases} \text{XP} = \text{XQ, by construction,} \\ \text{OX is common,} \\ \text{and PO} = \text{QO, being radii of equal circles;} \end{cases}$

$\therefore$ the $\angle$ OXP = the $\angle$ OXQ. *Theor.* 7.

And these being adjacent angles, each is a right angle;
that is, XO is perp. to AB.

Obs. If the point X is near one end of AB, one or other of the alternative constructions on the next page should be used.

PROBLEM 3. SECOND METHOD.



Construction. Take any point C outside AB.

With centre C, and radius CX, draw a circle cutting AB at D.

Join DC, and produce it to meet the circumference of the circle at O.

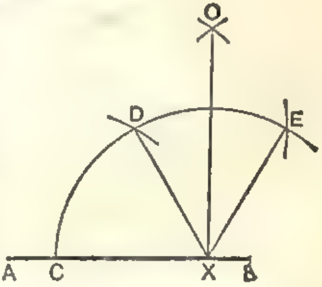
Join XO.

Then XO is perp. to AB.

Proof. Join CX.

Because CO = CX; $\therefore$ the $\angle$ CXO = the $\angle$ COX;
and because CD = CX; $\therefore$ the $\angle$ CXD = the $\angle$ CDX.
$\therefore$ the whole $\angle$ DXO = the $\angle$ XOD + the $\angle$ XDO
$= \frac{1}{2}$ of 180°
$=$ 90°.
$\therefore$ XO is perp. to AB.

PROBLEM 3. THIRD METHOD.



Construction. With centre X and any radius, draw the arc CDE, cutting AB at C.

With centre C, and with the same radius, draw an arc, cutting the first arc at D.

With centre D, and with the same radius, draw an arc, cutting the first arc at E.

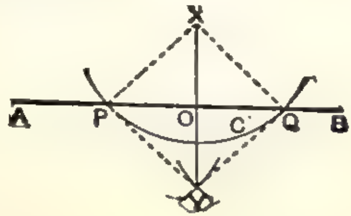
Bisect the $\angle$ DXE by XO *Prob.* 1.

Then XO is perp. to AB.

Proof. Each of the $\angle$ˢ CXD, DXE may be proved to be 60°;
and the $\angle$ DXO is half of the $\angle$ DXE;
$\therefore$ the $\angle$ CXO is 90°.
That is, XO is perp. to AB.

PROBLEM 4.

To draw a straight line perpendicular to a given straight line from a given external point.



Let X be the given external point from which a perpendicular is to be drawn at AB.

Construction. Take any point C on the side of AB remote from X.

With centre X, and radius XC, draw an arc to cut AB at P and Q.

With centres P and Q, and radius PX, draw arcs cutting at Y, on the side of AB opposite to X.

Join XY cutting AB at O.

Then XO is perp. to AB.

Proof. Join PX, QX, PY, QY,

In the $\triangle^s$ PXY, QXY,

because $\begin{cases} PX = QX, \text{ being radii of a circle,} \\ PY = QY, \text{ for the same reason,} \\ \text{and } XY \text{ is common;} \end{cases}$

$\therefore$ the $\angle PXY =$ the $\angle QXY$. *Theor.* 7.

Again, in the $\triangle^s$ PXO, QXO,

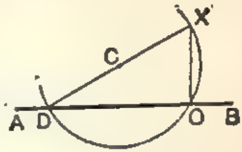
because $\begin{cases} PX = QX, \\ XO \text{ is common,} \\ \text{and the } \angle PXO = \text{the } \angle QXO; \end{cases}$

$\therefore$ the $\angle XOP =$ the $\angle XOQ$. *Theor.* 4.

And these being adjacent angles, each is a right angle, that is, XO is perp. to AB.

Obs. When the point X is nearly opposite one end of AB, one or other of the alternative constructions given below should be used.

PROBLEM 4. SECOND METHOD.



Construction. Take any point D in AB. Join DX, and bisect it at C.

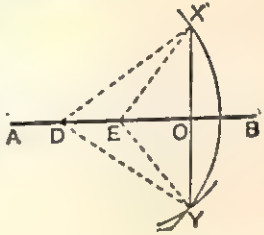
With centre C, and radius CX, draw a circle cutting AB at D and O.

Join XO.

Then XO is perp. to AB.

For, as in Problem 3, Second Method, the ∠XOD is a right angle.

PROBLEM 4. THIRD METHOD.



Construction. Take any two points D and E in AB.

With centre D, and radius DX, draw an arc of a circle, on the side of AB opposite to X.

With centre E, and radius EX, draw another arc cutting the former at Y.

Join XY, cutting AB at O.

Then XO is perp. to AB.

(i) Prove the △ˢ XDE, YDE equal in all respects by Theorem 7,

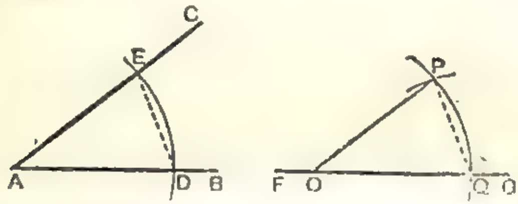
so that the ∠XDE = the ∠YDE.

(ii) Hence prove the △ˢ XDO, YDO equal in all respects by Theorem 4, so that the adjacent ∠ˢ DOX, DOY are equal.

That is, XO is perp. to AB.

PROBLEM 5.

At a given point in a given straight line to make an angle equal to a given angle.



Let BAC be the given angle, and FG the given straight line; and let O be the point at which an angle is to be made equal to the $\angle$ BAC.

Construction. With centre A, and with any radius, draw an arc cutting AB and AC at D and E.

With centre O, and with the same radius, draw an arc cutting FG at Q.

With centre Q, and with radius DE, draw an arc cutting the former arc at P.

Join OP.
Then POQ is the required angle.

Proof. Join ED, PQ.

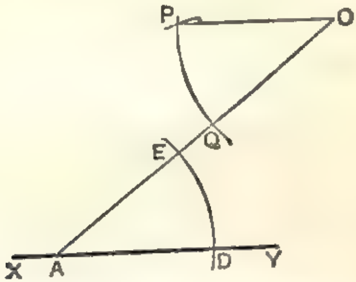
In the $\triangle$s POQ, EAD,

because $\begin{cases} \text{OP} = \text{AE, being radii of equal circles,} \\ \text{OQ} = \text{AD, for the same reason,} \\ \text{PQ} = \text{ED, by construction;} \end{cases}$

$\therefore$ the triangles are equal in all respects;
so that the $\angle$ POQ = the $\angle$ EAD. *Theor. 7*

Problem 6.

Through a given point to draw a straight line parallel to a given straight line.



Let XY be the given straight line, and O the given point, through which a straight line is to be drawn par[l] to XY.

Construction. In XY take any point A, and join OA.
Using the construction of Problem 5, at the point O in the line AO make the $\angle$AOP equal to the $\angle$OAY and alternate to it.

Then OP is parallel to XY.

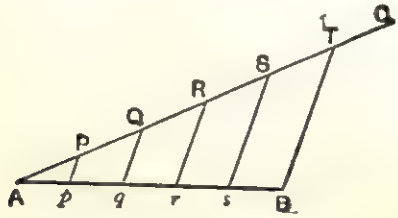
Proof. Because AO, meeting the straight lines OP, XY, makes the alternate $\angle$[s] POA, OAY equal;

$\therefore$ OP is par[l] to XY.

*** *The constructions of Problems 3, 4, and 6 are not usually followed in practical applications. Parallels and perpendiculars may be more quickly drawn by the aid of set squares.* (*See* LESSONS IN EXPERIMENTAL GEOMETRY, *pp.* 36, 42.)

PROBLEM 7.

To divide a given straight line into any number of equal parts.



Let AB be the given straight line, and suppose it is required to divide it into *five* equal parts.

Construction. From A draw AC, a straight line of unlimited length, making any angle with AB.

From AC mark off *five* equal parts of *any* length, AP, PQ, QR, RS, ST.

Join TB; and through P, Q, R, S draw par^{ls} to TB, meeting AB in *p*, *q*, *r*, *s*.

Then since the par^{ls} P*p*, Q*q*, R*r*, S*s*, TB cut off five equal parts from AT, they also cut off five equal parts from AB. (Theorem 22.)

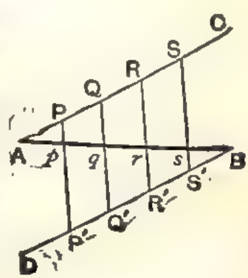
SECOND METHOD.

From A draw AC at any angle with AB, and on it mark off *four* equal parts AP, PQ, QR, RS, of any length.

From B draw BD par^{l} to AC, and on it mark off BS′, S′R′, R′Q′, Q′P′, each equal to the parts marked on AC.

Join PP′, QQ′, RR′, SS′ meeting AB in *p*, *q*, *r*, *s*. Then AB is divided into five equal parts at these points.

[Prove by Theorems 20 and 22.]



EXERCISES ON LINES AND ANGLES.

(Graphical Exercises.)

1. Construct (with ruler and compasses only) an angle of 60°.

By repeated bisection divide this angle into *four* equal parts.

2. By means of Exercise 1, *trisect a right angle*; that is, divide it into *three* equal parts.

Bisect each part, and hence shew how to trisect an angle of 45°.

[No construction is known for exactly trisecting *any* angle.]

3. Draw a line 6·7 cm. long, and divide it into *five* equal parts. Measure one of the parts in inches (to the nearest hundredth), and verify your work by calculation. [1 cm. = 0·3937 inch.]

4. From a straight line 3·72" long, cut off *one seventh*. Measure the part in centimetres and the nearest millimetre, and verify your work by calculation.

5. At a point X in a straight line AB draw XP perpendicular to AB, making XP 1·8" in length. From P draw an oblique PQ, 3·0" long, to meet AB in Q. Measure XQ.

(Problems. State your construction, and give a theoretical proof.)

6. In a straight line XY find a point which is equidistant from two given points A and B.

When is this impossible?

7. In a straight line XY find a point which is equidistant from two intersecting lines AB, AC.

When is this impossible?

8. From a given point P draw a straight line PQ, making with a given straight line AB an angle of given magnitude.

9. From two given points P and Q on the same side of a straight line AB, draw two lines which meet in AB and make equal angles with it.

[*Construction.* From P draw PH perp. to AB, and produce PH to P′, making HP′ equal to PH. Join P′Q cutting AB at K. Join PK. Prove that PK, QK are the required lines.]

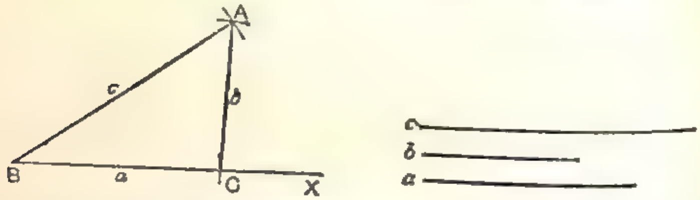
10. Through a given point P draw a straight line such that the perpendiculars drawn to it from two points A and B may be equal.

Is this always possible?

The Construction of Triangles.

Problem 8.

To draw a triangle having given the lengths of the three sides.



Let a, b, c be the lengths to which the sides of the required triangle are to be equal.

Construction. Draw any straight line BX, and cut off from it a part BC equal to a.

With centre B, and radius c, draw an arc of a circle.

With centre C, and radius b, draw a second arc cutting the first at A.

Join AB, AC.

Then ABC is the required triangle, for by construction the sides BC, CA, AB are equal to a, b, c respectively.

Obs. The three data a, b, c may be understood in two ways: either as three actual lines to which the sides of the triangle are to be equal, or as three *numbers* expressing the lengths of those lines in terms of inches, centimetres, or some other linear unit.

Notes. (i) In order that the construction may be possible it is necessary that any two of the given sides should be together greater than the third side (Theorem 11); for otherwise the arcs drawn from the centres B and C would not cut.

(ii) The arcs which cut at A would, if continued, cut again on the other side of BC. Thus the construction gives two triangles on opposite sides of a common base.

ON THE CONSTRUCTION OF TRIANGLES.

It has been seen (page 50) that to prove two triangles identically equal, *three* parts of one must be given equal to the corresponding parts of the other (though *any* three parts do not necessarily serve the purpose). This amounts to saying that *to determine the shape and size of a triangle we must know* three *of its parts*: or, in other words,

To construct a triangle three independent data are required.

For example, we may construct a triangle

(i) When *two sides* (b, c) and the *included angle* (A) are given.

The method of construction in this case is obvious.

(ii) When *two angles* (A, B) and one side (a) are given.

Here, since A and B are given, we at once know C;

for $A+B+C=180°$.

Hence we have only to draw the base equal to a, and at its ends make angles equal to B and C; for we know that the remaining angle must necessarily be equal to A.

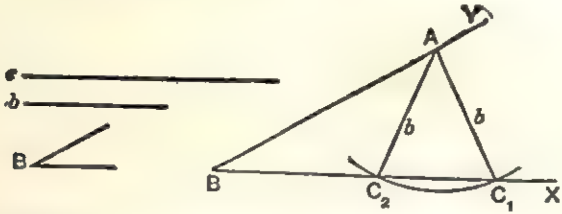
(iii) If the *three angles* A, B, C are given (and no side), the problem is indeterminate, that is, the number of solutions is unlimited.

For if at the ends of *any* base we make angles equal to B and C, the third angle is equal to A.

This construction is indeterminate, because the three data are not *independent*, the third following necessarily from the other two.

Problem 9.

To construct a triangle having given two sides and an angle opposite to one of them.



Let b, c be the given sides and B the given angle.

Construction. Take any straight line BX, and at B make the $\angle$XBY equal to the given $\angle$B.

From BY cut off BA equal to c.

With centre A, and radius b, draw an arc of a circle.

If this arc cuts BX in two points C_1 and C_2, both on the same side of B, both the $\triangle^s$ ABC_1, ABC_2 satisfy the given conditions.

This double solution is known as the **Ambiguous Case**, and will occur when b is less than c but greater than the perp. from A on BX.

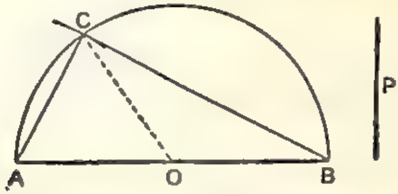
EXERCISE.

Draw figures to illustrate the nature and number of solutions in the following cases:

(i) When b is greater than c.
(ii) When b is equal to c.
(iii) When b is equal to the perpendicular from A on BX.
(iv) When b is less than this perpendicular.

PROBLEM 10.

To construct a right-angled triangle having given the hypotenuse and one side.



Let AB be the hypotenuse and P the given side.

Construction. Bisect AB at O; and with centre O, and radius OA, draw a semicircle.

With centre A, and radius P, draw an arc to cut the semicircle at C.

Join AC, BC.

Then ABC is the required triangle.

Proof. Join OC.

Because OA = OC;
the ∠ OCA = the ∠ OAC.

And because OB = OC;
∴ the ∠ OCB = the ∠ OBC.

∴ the whole ∠ ACB = the ∠ OAC + the ∠ OBC
$= \frac{1}{2}$ of 180° *Theor.* 16.
= 90°.

ON THE CONSTRUCTION OF TRIANGLES.

(*Graphical Exercises.*)

1. Draw a triangle whose sides are 7·5 cm., 6·2 cm., and 5·3 cm.

Draw and measure the perpendiculars dropped on these sides from the opposite vertices.

[N.B. The perpendiculars, if correctly drawn, will meet at a point, as will be seen later. See page 207.]

2. Draw a triangle ABC, having given $a = 3{\cdot}00''$, $b = 2{\cdot}50''$, $c = 2{\cdot}75''$.

Bisect the angle A by a line which meets the base at X. Measure BX and XC (to the nearest hundredth of an inch); and hence calculate the value of $\frac{\mathrm{BX}}{\mathrm{CX}}$ to two places of decimals. Compare your result with the value of $\frac{c}{b}$.

3. Two sides of a triangular field are 315 yards and 260 yards, and the included angle is known to be 39°. Draw a plan (1 inch to 100 yards) and find by measurement the length of the remaining side of the field.

4. ABC is a triangular plot of ground, of which the base BC is 75 metres, and the angles at B and C are 47° and 68° respectively. Draw a plan (scale 1 cm. to 10 metres). Write down without measurement the size of the angle A; and by measuring the plan, obtain the approximate lengths of the other sides of the field; also the perpendicular drawn from A to BC.

5. A yacht on leaving harbour steers N.E. sailing 9 knots an hour. After 20 minutes she goes about, steering N.W. for 35 minutes and making the same average speed as before. How far is she now from the harbour, and what course (approximately) must she set for the run home? Obtain your results from a chart of the whole course, scale 2 cm. to 1 knot.

6. Draw a right-angled triangle, given that the hypotenuse $c = 10{\cdot}6$ cm. and one side $a = 5{\cdot}6$ cm. Measure the third side b; and find the value of $\sqrt{c^2 - a^2}$. Compare the two results.

7. Construct a triangle, having given the following parts: $\mathrm{B} = 34°$, $b = 5{\cdot}5$ cm., $c = 8{\cdot}5$ cm. Shew that there are two solutions. Measure the two values of a, and also of C, and shew that the latter are supplementary.

8. In a triangle ABC, the angle $\mathrm{A} = 50°$, and $b = 6{\cdot}5$ cm. Illustrate by figures the cases which arise in constructing the triangle, when (i) $a = 7$ cm. (ii) $a = 6$ cm. (iii) $a = 5$ cm. (iv) $a = 4$ cm.

9. Two straight roads, which cross at right angles at A, are carried over a straight canal by bridges at B and C. The distance between the bridges is 461 yards, and the distance from the crossing A to the bridge B is 261 yards. Draw a plan, and by measurement of it ascertain the distance from A to C.

(*Problems. State your construction, and give a theoretical proof.*)

10. Draw an isosceles triangle on a base of 4 cm., and having an altitude of 6·2 cm. *Prove* the two sides equal, and measure them to the nearest millimetre.

11. Draw an isosceles triangle having its vertical angle equal to a given angle, and the perpendicular from the vertex on the base equal to a given straight line.

Hence draw an equilateral triangle in which the perpendicular from one vertex on the opposite side is 6 cm. Measure the length of a side to the nearest millimetre.

12. Construct a triangle ABC in which the perpendicular from A on BC is 5·0 cm., and the sides AB, AC are 5·8 cm. and 9·0 cm. respectively. Measure BC.

13. Construct a triangle ABC having the angles at B and C equal to two given angles L and M, and the perpendicular from A on BC equal to a given line P.

14. Construct a triangle ABC (without protractor) having given two angles B and C and the side b.

15. On a given base construct an isosceles triangle having its vertical angle equal to the given angle L.

16. Construct a right-angled triangle, having given the length of the hypotenuse c, and the sum of the remaining sides a and b.

If $c=5{\cdot}3$ cm., and $a+b=7{\cdot}3$ cm., find a and b graphically; and calculate the value of $\sqrt{a^2+b^2}$.

17. Construct a triangle having given the perimeter and the angles at the base. For example, $a+b+c=12$ cm., $B=70°$, $C=80°$

18. Construct a triangle ABC from the following data:

$$a=6{\cdot}5 \text{ cm.}, \quad b+c=10 \text{ cm.}, \quad \text{and } B=60°.$$

Measure the lengths of b and c.

19. Construct a triangle ABC from the following data:

$$a=7 \text{ cm.}, \quad c-b=1 \text{ cm.}, \quad \text{and } B=55°.$$

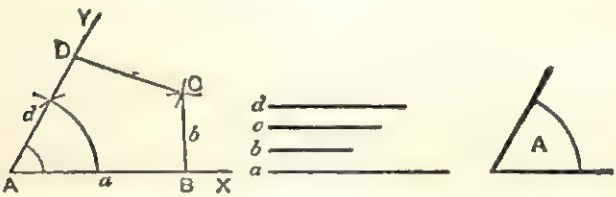
Measure the lengths of b and c.

THE CONSTRUCTION OF QUADRILATERALS.

It has been shewn that the shape and size of a triangle are completely determined when the lengths of its three sides are given. A quadrilateral, however, is not completely determined by the lengths of its four sides. From what follows it will appear that *five* independent data are required to construct a quadrilateral.

PROBLEM 11.

To construct a quadrilateral, given the lengths of the four sides, and one angle.



Let a, b, c, d be the given lengths of the sides, and A the angle between the sides equal to a and d.

Construction. Take any straight line AX, and cut off from it AB equal to a.

Make the ∠ BAY equal to the ∠ A.

From AY cut off AD equal to d.

With centre D, and radius c, draw an arc of a circle.

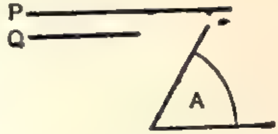
With centre B and radius b, draw another arc to cut the former at C.

Join DC, BC.

Then ABCD is the required quadrilateral; for by construction the sides are equal to a, b, c, d, and the ∠ DAB is equal to the given angle.

PROBLEM 12.

To construct a parallelogram having given two adjacent sides and the included angle.



Let P and Q be the two given sides, and A the given angle.

Construction 1. (*With ruler and compasses.*) Take a line AB equal to P; and at A make the ∠ BAD equal to the ∠ A, and make AD equal to Q.

With centre D, and radius P, draw an arc of a circle.

With centre B, and radius Q, draw another arc to cut the former at C.

Then ABCD is the required par^{m}.

Proof. Join DB.

In the △^{s} DCB, BAD,

because $\begin{cases} DC = BA, \\ CB = AD, \\ \text{and } DB \text{ is common:} \end{cases}$

∴ the ∠ CDB = the ∠ ABD; *Theor.* 7.

and these are alternate angles,

∴ DC is par^{l} to AB.

Also DC = AB;

∴ DA and BC are also equal and parallel. *Theor.* 20.

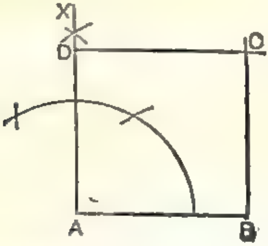
∴ ABCD is a par^{m}.

Construction 2. (*With set squares.*) Draw AB and AD as before; then with set squares through D draw DC par^{l} to AB, and through B draw BC par^{l} to AD.

By construction, ABCD is a par^{m} having the required parts.

Problem 13.

To construct a square on a given side.



Let AB be the given side.

Construction 1. (*With ruler and compasses.*) At A draw AX perp. to AB, and cut off from it AD equal to AB.

With B and D as centres, and with radius AB, draw two arcs cutting at C.

Join BC, DC.

Then ABCD is the required square.

Proof. As in Problem 12, ABCD may be shewn to be a par$^{\text{m}}$. And since the ∠BAD is a right angle, the figure is a rectangle. Also, by construction all its sides are equal.

∴ ABCD is a square.

Construction 2. (*With set squares.*) At A draw AX perp. to AB, and cut off from it AD equal to AB.

Through D draw DC par$^{\text{l}}$ to AB, and through B draw BC par$^{\text{l}}$ to AD meeting DC in C.

Then, by construction, ABCD is a rectangle. [Def. 3, page 56.]

Also it has the two adjacent sides AB, AD equal.

∴ it is a square.

EXERCISES.

ON THE CONSTRUCTION OF QUADRILATERALS.

1. Draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure.

Ascertain (without measurement) the number of degrees in each angle, giving a reason for your answer.

2. Draw a square on a side of 2.5 inches. Prove theoretically that its diagonals are equal; and by measuring the diagonals to the nearest hundredth of an inch test the correctness of your drawing.

3. Construct a square on a diagonal of 3·0″, and measure the lengths of each side. Obtain the average of your results.

4. Draw a parallelogram ABCD, having given that one side AB = 5·5 cm., and the diagonals AC, BD are 8 cm., and 6 cm. respectively. Measure AD.

5. The diagonals of a certain quadrilateral are equal, (each 6·0 cm.), and they bisect one another at an angle of 60°. Shew that *five* independent data are here given.

Construct the quadrilateral. Name its species; and give a *formal* proof of your answer. Measure the perimeter. If the angle between the diagonals were increased to 90°, by how much per cent. would the perimeter be increased?

6. In a quadrilateral ABCD,

AB = 5·6 cm., BC = 2·5 cm., CD = 4·0 cm., and DA = 3·3 cm.

Shew that the shape of the quadrilateral is not settled by these data.

Draw the quadrilateral when (i) A = 30°, (ii) A = 60°. Why does the construction fail when A = 100°?

Determine graphically the least value of A for which the construction fails.

7. Shew how to construct a quadrilateral, having given the lengths of the four sides and of one diagonal. What conditions must hold among the data in order that the problem may be possible?

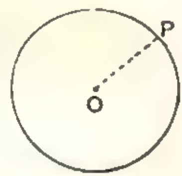
Illustrate your method by constructing a quadrilateral ABCD, when

(i) AB = 3·0″, BC = 1·7″, CD = 2·5″, DA = 2·8″, and the diagonal BD = 2·6″. Measure AC.

(ii) AB = 3·6 cm., BC = 7·7 cm., CD = 6·8 cm., DA = 5·1 cm., and the diagonal AC = 8·5 cm. Measure the angles at B and D.

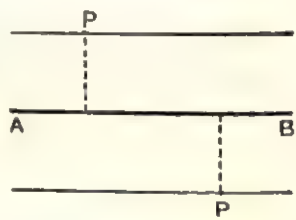
LOCI.

DEFINITION. The locus of a point is the path traced out by it when it moves in accordance with some given law.



Example 1. Suppose the point P to move so that its distance from a fixed point O is constant (say 1 centimetre).

Then the locus of P is evidently the circumference of a circle whose centre is O and radius 1 cm.



Example 2. Suppose the point P moves at a constant distance (say 1 cm.) from a fixed straight line AB.

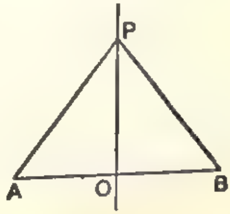
Then the locus of P is one or other of two straight lines parallel to AB, on either side, and at a distance of 1 cm. from it.

Thus the locus of a point, moving under some given condition, consists of the line or lines to which the point is thereby restricted; provided that the condition is satisfied by every point on such line or lines, and by no other.

When we find a series of points which satisfy the given law, and through which therefore the moving point must pass, we are said to plot the locus of the point.

PROBLEM 14.

To find the locus of a point P *which moves so that its distances from two fixed points* A *and* B *are always equal to one another.*



Here the point P moves through all positions in which PA = PB;
$\therefore$ one position of the moving point is at O the middle point of AB.

Suppose P to be *any* other position of the moving point: that is, let PA = PB.

Join OP.

Then in the $\triangle^s$ POA, POB,

because $\begin{cases} \text{PO is common,} \\ \text{OA} = \text{OB,} \\ \text{and PA} = \text{PB, by hypothesis;} \end{cases}$

$\therefore$ the $\angle$ POA = the $\angle$ POB. *Theor.* 7.

Hence PO is perpendicular to AB.

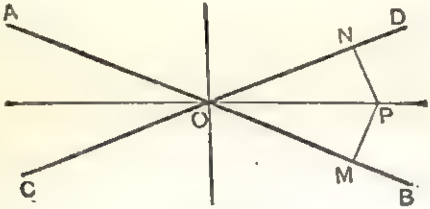
That is, *every point* P *which is equidistant from* A *and* B *lies on the straight line bisecting* AB *at right angles.*

Likewise it may be proved that every point on the perpendicular through O is equidistant from A and B.

This line is therefore the required locus.

PROBLEM 15.

To find the locus of a point P *which moves so that its perpendicular distances from two given straight lines* AB, CD *are equal to one another.*



Let P be any point such that the perp. PM = the perp. PN.

Join P to O, the intersection of AB, CD.

Then in the $\triangle^s$ PMO, PNO,

because $\begin{cases}\text{the } \angle^s \text{ PMO, PNO are } \textit{right angles}, \\ \text{the hypotenuse OP is common,} \\ \text{and one side PM = one side PN;}\end{cases}$

∴ the triangles are equal in all respects; *Theor.* 18
so that the ∠ POM = the ∠ PON.

Hence, if P lies within the ∠ BOD, it must be on the bisector of that angle;

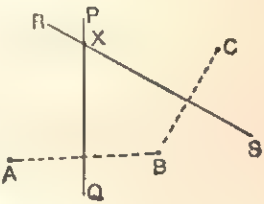
and, if P is within the ∠ AOD, it must be on the bisector of that angle.

It follows that *the required locus is the pair of lines which bisect the angles between* AB *and* CD.

INTERSECTION OF LOCI.

The method of Loci may be used to find the position of a point which is subject to two conditions. For corresponding to each condition there will be a locus on which the required point must lie. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

EXAMPLE 1. *To find a point equidistant from three given points* A, B, C, *which are not in the same straight line.*

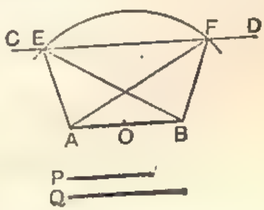


(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles.

(ii) Similarly, the locus of points equidistant from B and C is the straight line RS which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, X the point of intersection of PQ and RS will be equidistant from A, B, and C.

EXAMPLE 2. *To construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.*



Let AB be the given base, and P and Q the lengths of the altitude and median respectively.

Then the triangle is known if its *vertex* is known.

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P: *then the required vertex must lie on* CD.

(ii) Again, from O the middle point of AB as centre, with radius equal to Q, describe a circle:

then the required vertex must lie on this circle.

Hence any points which are common to CD and the circle, satisfy both the given conditions: that is to say, if CD intersect the circle in E, F, each of the points of intersection might be the vertex of the required triangle. This supposes the length of the median Q to be greater than the altitude.

It may happen that the data of the problem are so related to one another that the resulting loci do not intersect. In this case the problem is impossible.

Obs. In examples on the Intersection of Loci the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible; and he must observe that if under certain relations *two* solutions are possible, and under other relations no solution exists, there will always be some *intermediate* relation under which the two solutions combine in a single solution.

EXAMPLES ON LOCI.

1. Find the locus of a point which moves so that its distance (measured radially) from the circumference of a given circle is constant.

2. A point P moves along a straight line RQ; find the position in which it is equidistant from two given points A and B.

3. A and B are two fixed points within a circle: find points on the circumference equidistant from A and B. How many such points are there?

4. A point P moves along a straight line RQ; find the position in which it is equidistant from two given straight lines AB and CD.

5. A and B are two fixed points 6 cm. apart. Find by the method of loci two points which are 4 cm. distant from A, and 5 cm. from B.

6. AB and CD are two given straight lines. Find points 3 cm. distant from AB, and 4 cm. from CD. How many solutions are there?

7. A straight rod of given length slides between two straight rulers placed at right angles to one another.

Plot the locus of its middle point; and shew that this locus is the fourth part of the circumference of a circle. [See Problem 10.]

8. On a given base as hypotenuse right-angled triangles are described. Find the locus of their vertices.

9. A is a fixed point, and the point X moves on a fixed straight line BC.

Plot the locus of P, the middle point of AX; and *prove* the locus to be a straight line parallel to BC.

10. A is a fixed point, and the point X moves on the circumference of a given circle.

Plot the locus of P, the middle point of AX; and *prove* that this locus is a circle. [See Ex. 3, p. 64.]

11. AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B. If BX revolve about B, find the locus of the middle point of AX.

12. Two straight lines OX, OY cut at right angles, and from P, a point within the angle XOY, perpendiculars PM, PN are drawn to OX, OY respectively. Plot the locus of P when

(i) $PM + PN$ is constant (=6 cm., say):

(ii) $PM - PN$ is constant (=3 cm., say).

And in each case give a theoretical proof of the result you arrive at experimentally.

13. Two straight lines OX, OY intersect at right angles at O; and from a movable point P perpendiculars PM, PN are drawn to OX, OY.

Plot (without proof) the locus of P, when

(i) $PM = 2PN$;

(ii) $PM = 3PN$.

14. Find a point which is at a given distance from a given point, and is equidistant from two given parallel straight lines.

When does this problem admit of two solutions, when of one only, and when is it impossible?

15. S is a fixed point 2 inches distant from a given straight line MX. Find two points which are $2\frac{3}{4}$ inches distant from S, and also $2\frac{3}{4}$ inches distant from MX.

16. Find a series of points equidistant from a given point S and a given straight line MX. Draw a curve freehand passing through all the points so found.

17. On a given base construct a triangle of given altitude, having its vertex on a given straight line.

18. Find a point equidistant from the three sides of a triangle.

19. Two straight lines OX, OY cut at right angles; and Q and R are points in OX and OY respectively. Plot the locus of the middle point of QR, when

(i) $OQ + OR =$ constant.

(ii) $OQ - OR =$ constant.

20. S and S′ are two fixed points. Find a series of points P such that

(i) $SP + S'P =$ constant (say 3·5 inches).

(ii) $SP - S'P =$ constant (say 1·5 inch).

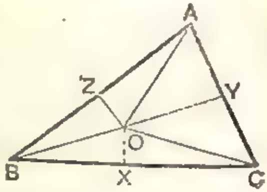
In each case draw a curve freehand passing through all the points so found.

ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

I. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*

Let ABC be a △, and X, Y, Z the middle points of its sides.

From Z and Y draw perps. to AB, AC, meeting at O. *Join* OX.



It is required to prove that OX *is perp. to* BC.

Join OA, OB, OC.

Proof. Because YO bisects AC at right angles,
∴ it is the locus of points equidistant from A and C;
∴ OA=OC.

Again, because ZO bisects AB at right angles,
∴ it is the locus of points equidistant from A and B;
∴ OA=OB.

Hence OB=OC.
∴ O is on the locus of points equidistant from B and C;
that is, OX is perp. to BC.

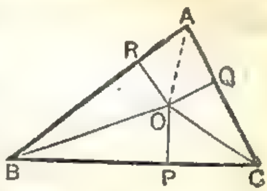
Hence the perpendiculars from the mid-points of the sides meet at O.

Q.E.D.

II. *The bisectors of the angles of a triangle are concurrent.*

Let ABC be a △. Bisect the ∠ˢ ABC, BCA by straight lines which meet at O.

Join AO.



It is required to prove that AO *bisects the* ∠BAC.

From O draw OP, OQ, OR perp. to the sides of the △.

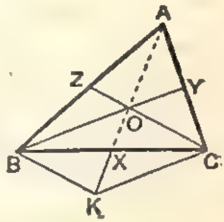
Proof. Because BO bisects the ∠ABC,
∴ it is the locus of points equidistant from BA and BC;
∴ OP=OR.

Similarly CO is the locus of points equidistant from BC and CA;
∴ OP=OQ.

Hence OR=OQ.
∴ O is on the locus of points equidistant from AB and AC;
that is, OA is the bisector of the ∠BAC.

Hence the bisectors of the angles meet at O. Q.E.D.

III. *The medians of a triangle are concurrent.*



Let ABC be a $\triangle$.

Let BY and CZ be two of its medians, and let them intersect at O.

Join AO,

and produce it to meet BC in X.

It is required to shew that AX *is the remaining median of the* $\triangle$.

Through C draw CK parallel to BY;

produce AX to meet CK at K.

Join BK.

Proof. In the $\triangle$AKC,

because Y is the middle point of AC, and YO is parallel to CK,

$\therefore$ O is the middle point of AK. *Theor.* 22.

Again in the $\triangle$ABK,

since Z and O are the middle points of AB, AK,

$\therefore$ ZO is parallel to BK,

that is, OC is parallel to BK,

$\therefore$ the figure BKCO is a par^{m}.

But the diagonals of a par^{m} bisect one another;

$\therefore$ X is the middle point of BC.

That is, AX is a median of the $\triangle$.

Hence the three medians meet at the point O. Q.E.D.

Definition. The point of intersection of the medians is called the centroid of the triangle.

Corollary. *The three medians of a triangle cut one another at a point of trisection, the greater segment in each being towards the angular point.*

For in the above figure it has been proved that

AO = OK,

also that OX is half of OK;

$\therefore$ OX is half of OA:

that is, OX is one third of AX.

Similarly OY is one third of BY,

and OZ is one third of CZ. Q.E.D.

By means of this Corollary it may be shewn that in any triangle the shorter median bisects the greater side.

Note. It will be proved hereafter that *the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

MISCELLANEOUS PROBLEMS.

(*A theoretical proof is to be given in each case.*)

1. A is a given point, and BC a given straight line. From A draw a straight line to make with BC an angle equal to a given angle X.

How many such lines can be drawn?

2. Draw the bisector of an angle AOB, without using the vertex O in your construction.

3. P is a given point within the angle AOB. Draw through P a straight line terminated by OA and OB, and bisected at P.

4. OA, OB, OC are three straight lines meeting at O. Draw a transversal terminated by OA and OC, and bisected by OB.

5. Through a given point A draw a straight line so that the part intercepted between two given parallels may be of given length.

When does this problem admit of two solutions? When of only one? And when is it impossible?

6. In a triangle ABC inscribe a rhombus having one of its angles coinciding with the angle A.

7. Use the properties of an equilateral triangle to trisect a given straight line.

(*Construction of Triangles.*)

8. Construct a triangle, having given

(i) The middle points of the three sides.

(ii) The lengths of two sides and of the median which bisects the third side.

(iii) The lengths of one side and the medians which bisect the other two sides.

(iv) The lengths of the three medians.

PART II.

ON AREAS.

Definitions.

1. The **altitude** (or **height**) of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

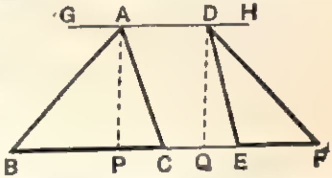
2. The **altitude** (or **height**) of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base.

Note. It is clear that *parallelograms or triangles which are between the same parallels have the same altitude.*

For let AP and DQ be the altitudes of the $\triangle^s$ ABC, DEF, which are between the same parallels BF, GH.

Then the fig. APQD is evidently a rectangle;

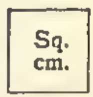
$\therefore$ AP = DQ.



3. The **area** of a figure is the amount of surface contained within its bounding lines.

4. A **square inch** is the area of a square drawn on a side one inch in length.

5. Similarly a **square centimetre** is the area of a square drawn on a side one centimetre in length.

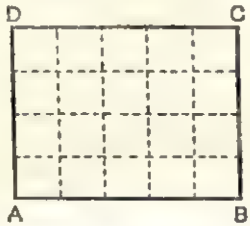


The terms *square yard, square foot, square metre* are to be understood in the same sense.

6. Thus the **unit of area** is the area of a square on a side of unit length.

THEOREM 23.

Area of a rectangle. *If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area.*



Let ABCD represent a rectangle whose length AB is 5 feet, and whose breadth AD is 4 feet.

Divide AB into 5 equal parts, and BC into 4 equal parts, and through the points of division of each line draw parallels to the other.

The rectangle ABCD is now divided into compartments, each of which represents one square foot.

Now there are 4 rows, each containing 5 squares,

$\therefore$ the rectangle contains 5×4 square feet.

Similarly, if the length $= a$ linear units, and the breadth $= b$ linear units

the rectangle contains **ab** *units of* **area.**

And if each side of a square $= a$ linear units,

the square contains $\mathbf{a^2}$ *units of* **area.**

These statements may be thus abridged:

the area of a rectangle = length × breadth ....(i),

the area of a square = (side)² ....(ii).

Q.E.D.

COROLLARIES. (i) *Rectangles which have equal lengths and equal breadths have equal areas.*

(ii) *Rectangles which have equal areas and equal lengths have also equal breadths.*

NOTATION.

The rectangle ABCD is said to be contained by AB, AD; for these adjacent sides fix its size and shape.

A rectangle whose adjacent sides are AB, AD is denoted by *rect.* AB, AD, or simply $AB \times AD$.

A square drawn on the side AB is denoted by *sq. on* AB, or AB^2.

EXERCISES.

(On Tables of Length and Area.)

1. Draw a figure to shew *why*

(i) 1 sq. yard $= 3^2$ sq. feet.
(ii) 1 sq. foot $= 12^2$ sq. inches.
(iii) 1 sq. cm. $= 10^2$ sq. mm.

2. Draw a figure to shew that the square on a straight line is four times the square on half the line.

3. Use squared paper to shew that the square on $1'' = 10^2$ times the square on $0{\cdot}1''$.

4. If $1''$ represents 5 miles, what does an area of 6 square inches represent?

EXTENSION OF THEOREM 23.

The proof of Theorem 23 here given supposes that the length and breadth of the given rectangle are expressed by *whole numbers*; but the formula holds good when the length and breadth are fractional.

This may be illustrated thus:

Suppose the length and breadth are 3·2 cm. and 2·4 cm.; we shall shew that the area is $(3{\cdot}2 \times 2{\cdot}4)$ sq. cm.

For
$$\text{length} = 3{\cdot}2 \text{ cm.} = 32 \text{ mm.}$$
$$\text{breadth} = 2{\cdot}4 \text{ cm.} = 24 \text{ mm.}$$
$$\therefore \text{ area} = (32 \times 24) \text{ sq. mm.} = \frac{32 \times 24}{10^2} \text{ sq. cm.}$$
$$= (3{\cdot}2 \times 2{\cdot}4) \text{ sq. cm.}$$

EXERCISES.

(*On the Area of a Rectangle.*)

Draw on squared paper the rectangles of which the length (a) and breadth (b) are given below. Calculate the areas, and verify by the actual counting of squares.

1. $a=2''$, $b=3''$.
2. $a=1{\cdot}5''$, $b=4''$.
3. $a=0{\cdot}8''$, $b=3{\cdot}5''$.
4. $a=2{\cdot}5''$, $b=1{\cdot}4''$.
5. $a=2{\cdot}2''$, $b=1{\cdot}5''$.
6. $a=1{\cdot}6''$, $b=2{\cdot}1''$.

Calculate the areas of the rectangles in which

7. $a=18$ metres, $b=11$ metres.
8. $a=7$ ft., $b=72$ in.
9. $a=2{\cdot}5$ km., $b=4$ metres.
10. $a=\frac{1}{8}$ mile, $b=1$ inch.

11. The area of a rectangle is 30 sq. cm., and its length is 6 cm. Find the breadth. Draw the rectangle on squared paper; and verify your work by counting the squares.

12. Find the length of a rectangle whose area is 3·9 sq. in., and breadth 1·5″. Draw the rectangle on squared paper; and verify your work by counting the squares.

13. (i) When you treble the length of a rectangle without altering its breadth, how many times do you multiply the area?

(ii) When you treble both length and breadth, how many times do you multiply the area?

Draw a figure to illustrate your answers; and state a general rule.

14. In a plan of a rectangular garden the length and breadth are 3·6″ and 2·5″, one inch standing for 10 yards. Find the area of the garden.

If the area is increased by 300 sq. yds., the breadth remaining the same, what will the new length be? And how many inches will represent it on your plan?

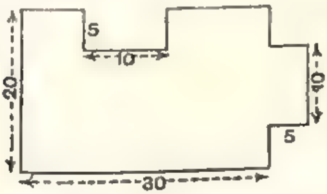
15. Find the area of a rectangular enclosure of which a plan (scale 1 cm. to 20 metres) measures 6·5 cm. by 4·5 cm.

16. The area of a rectangle is 1440 sq. yds. If in a plan the sides of the rectangle are 3·2 cm. and 4·5 cm., on what scale is the plan drawn?

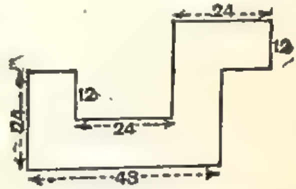
17. The area of a rectangular field is 52000 sq. ft. On a plan of this, drawn to the scale of 1″ to 100 ft., the length is 3·25″. What is the breadth?

Calculate the areas of the enclosures of which plans are given below. All the angles are right angles, and the dimensions are marked in feet.

18.

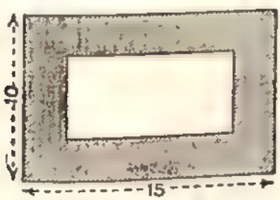


19.



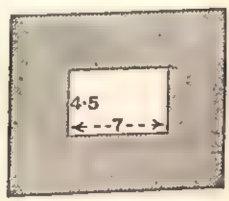
Calculate the areas represented by the shaded parts of the following plans. The dimensions are marked in feet.

20.



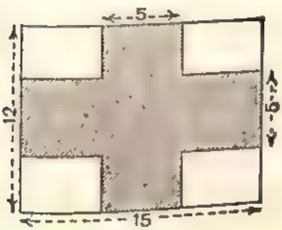
Width of shaded border uniform $2\frac{1}{2}$ ft.

21.

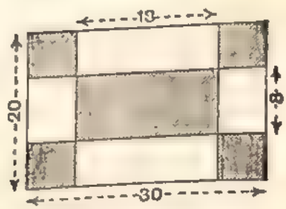


Width of shaded border uniform 4 ft.

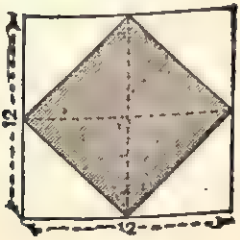
22.



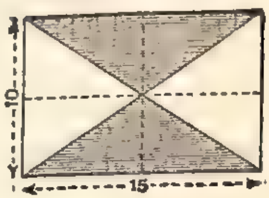
23.



24.

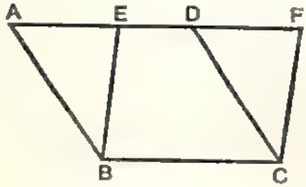


25.



THEOREM 24. [Euclid I. 35.]

Parallelograms on the same base and between the same parallels are equal in area.



Let the par^{ms} ABCD, EBCF be on the same base BC, and between the same par^{ls} BC, AF.

It is required to prove that

the par^m ABCD = *the par^m* EBCF *in area.*

Proof. In the $\triangle^s$ FDC, EAB,

because $\begin{cases} \text{DC} = \text{the opp. side AB}; & \textit{Theor. } 21. \\ \text{the ext. } \angle \text{FDC} = \text{the int. opp. } \angle \text{EAB}; & \textit{Theor. } 14. \\ \text{the int. } \angle \text{DFC} = \text{the ext. } \angle \text{AEB}; & \end{cases}$

$\therefore$ the $\triangle$ FDC = the $\triangle$ EAB. *Theor.* 17.

Now, if from the whole fig. ABCF the $\triangle$ FDC is taken, the remainder is the par^m ABCD.

And if from the whole fig. ABCF the $\triangle$ EAB is taken, the remainder is the par^m EBCF.

$\therefore$ these remainders are equal;

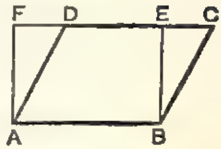
that is, the par^m ABCD = the par^m EBCF. Q.E.D.

EXERCISE.

In the above diagram the sides AD, EF overlap. Draw diagrams in which (i) these sides do not overlap; (ii) the ends E and D coincide.

Go through the proof with these diagrams, and ascertain if it applies to them without change.

The Area of a Parallelogram.



Let ABCD be a parallelogram, and ABEF the rectangle on the same base AB and of the same altitude BE. Then by Theorem 24,

area of par^m ABCD = area of rect. ABEF
= AB × BE
= *base × altitude.*

Corollary. Since the area of a parallelogram depends only on its base and altitude, it follows that

Parallelograms on equal bases and of equal altitudes are equal in area.

EXERCISES.

(Numerical and Graphical.)

1. Find the area of parallelograms in which
   (i) the base $=5\cdot5$ cm., and the height $=4$ cm.
   (ii) the base $=2\cdot4''$, and the height $=1\cdot5''$.

2. Draw a parallelogram ABCD having given AB $=2\frac{1}{2}''$, AD $=1\frac{1}{2}''$, and the $\angle$A $=65°$. Draw and measure the perpendicular from D on AB, and hence calculate the approximate area. Why *approximate*?

   Again calculate the area from the length of AD and the perpendicular on it from B. Obtain the average of the two results.

3. Two adjacent sides of a parallelogram are 30 metres and 25 metres, and the included angle is 50°. Draw a plan, 1 cm. representing 5 metres; and by measuring each altitude, make two independent calculations of the area. Give the average result.

4. The area of a parallelogram ABCD is 4·2 sq. in., and the base AB is 2·8″. Find the height. If AD $=2''$, draw the parallelogram.

5. Each side of a rhombus is 2″, and its area is 3·86 sq. in. Calculate an altitude. Hence draw the rhombus, and measure one of its acute angles

Theorem 25.

The Area of a Triangle. *The area of a triangle is half the area of the rectangle on the same base and having the same altitude.*

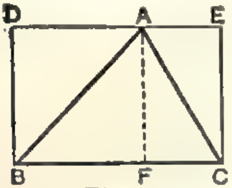


Fig. 1.

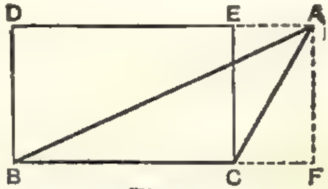


Fig. 2.

Let ABC be a triangle, and BDEC a rectangle on the same base BC and with the same altitude AF.

It is required to prove that the △ ABC *is half the rectangle* BDEC.

Proof. Since AF is perp. to BC, each of the figures DF, EF is a rectangle.

Because the diagonal AB bisects the rectangle DF,
∴ the △ ABF is half the rectangle DF.

Similarly, the △ AFC is half the rectangle FE.

∴ adding these results in Fig. 1, and taking the difference in Fig. 2,
the △ ABC is half the rectangle BDEC.

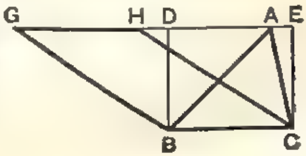
Q.E.D.

Corollary. *A triangle is half any parallelogram on the same base and between the same parallels.*

For the △ ABC is half the rect. BCED.

And the rect. BCED = any par^m^ BCHG on the same base and between the same par^ls^.

∴ the △ ABC is half the par^m^ BCHG.



THE AREA OF A TRIANGLE.

If BC and AF respectively contain a units and p units of length, the rectangle BDEC contains ap units of area.

$\therefore$ the area of the $\triangle$ ABC $= \frac{1}{2}ap$ units of area.

This result may be stated thus:

Area of a Triangle $= \frac{1}{2}$. *base* $\times$ *altitude.*

EXERCISES ON THE AREA OF A TRIANGLE.

(Numerical and Graphical.)

1. Calculate the areas of the triangles in which
   (i) the base $=24$ ft., the height $=15$ ft.
   (ii) the base $=4\cdot8''$, the height $=3\cdot5''$.
   (iii) the base $=160$ metres, the height $=125$ metres.

2. Draw triangles from the following data. In each case draw and measure the altitude with reference to a given side as base: hence calculate the approximate area.
   (i) $a=8\cdot4$ cm., $b=6\cdot8$ cm., $c=4\cdot0$ cm.
   (ii) $b=5\cdot0$ cm., $c=6\cdot8$ cm., $A=65°$.
   (iii) $a=6\cdot5$ cm., $B=52°$, $C=76°$.

3. ABC is a triangle *right-angled* at C; shew that its area $=\frac{1}{2}$BC $\times$ CA.
   Given $a=6$ cm., $b=5$ cm., calculate the area.
   Draw the triangle and measure the hypotenuse c; draw and measure the perpendicular from C on the hypotenuse; hence calculate the approximate area.
   Note the error in your approximate result, and express it as a percentage of the true value.

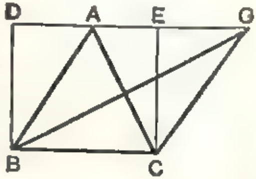
4. Repeat the whole process of the last question for a right-angled triangle ABC, in which $a=2\cdot8''$ and $b=4\cdot5''$; C being the right angle as before.

5. In a triangle, given
   (i) Area $=80$ sq. in., base $=1$ ft. 8 in.; calculate the altitude.
   (ii) Area $=10\cdot4$ sq. cm., altitude $=1\cdot6$ cm.; calculate the base.

6. Construct a triangle ABC, having given $a=3\cdot0''$, $b=2\cdot8''$, $c=2\cdot6''$. Draw and measure the perpendicular from A on BC; hence calculate the approximate area.

THEOREM 26. [Euclid I. 37.]

Triangles on the same base and between the same parallels (hence, of the same altitude) are equal in area.



Let the $\triangle^s$ ABC, GBC be on the same base BC and between the same par^{ls} BC, AG.

It is required to prove that
the $\triangle$ ABC = *the* $\triangle$ GBC *in area.*

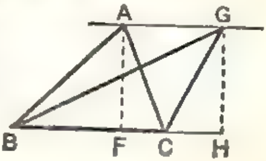
Proof. If BCED is the rectangle on the base BC, and between the same parallels as the given triangles,

the $\triangle$ ABC is half the rect. BCED; *Theor.* 25.
also the $\triangle$ GBC is half the rect. BCED;
$\therefore$ the $\triangle$ ABC = the $\triangle$ GBC. Q.E.D.

Similarly, *triangles on equal bases and of equal altitudes are equal in area.*

THEOREM 27. [Euclid I. 39.]

If two triangles are equal in area, and stand on the same base and on the same side of it, they are between the same parallels.



Let the $\triangle^s$ ABC, GBC, standing on the same base BC, be equal in area; and let AF and GH be their altitudes.

It is required to prove that AG *and* BC *are par^l.*

Proof. The $\triangle$ ABC is half the rectangle contained by BC and AF;

and the $\triangle$GBC is half the rectangle contained by BC and GH;

$\therefore$ the rect. BC, AF = the rect. BC, GH;
$\therefore$ AF = GH. *Theor.* 23, *Cor.* 2.
Also AF and GH are par^l;
hence AG and FH, that is BC, are par^l. Q.E.D.

EXERCISES ON THE AREA OF A TRIANGLE.

(*Theoretical.*)

1. ABC is a triangle and XY is drawn parallel to the base BC, cutting the other sides at X and Y. Join BY and CX; and shew that

(i) the $\triangle$ XBC = the $\triangle$ YBC;
(ii) the $\triangle$ BXY = the $\triangle$ CXY;
(iii) the $\triangle$ ABY = the $\triangle$ ACX.

If BY and CX cut at K, shew that

(iv) the $\triangle$ BKX = the $\triangle$ CKY.

2. Shew that a median of a triangle divides it into two parts of equal area.

How would you divide a triangle into *three* equal parts by straight lines drawn from its vertex?

3. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.

4. ABC is a triangle whose base BC is bisected at X. If Y is any point in the median AX, shew that

the $\triangle$ ABY = the $\triangle$ ACY in area.

5. ABCD is a parallelogram, and BP, DQ are the perpendiculars from B and D on the diagonal AC.

Shew that BP = DQ.

Hence if X is any point in AC, or AC produced,

prove (i) the $\triangle$ ADX = the $\triangle$ ABX;
(ii) the $\triangle$ CDX = the $\triangle$ CBX.

6. Prove by means of Theorems 26 and 27 that *the straight line joining the middle points of two sides of a triangle is parallel to the third side.*

7. The straight line which joins the middle points of the oblique sides of a trapezium is parallel to each of the parallel sides.

8. ABCD is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD.

9. If ABCD is a parallelogram, and X, Y any points in DC and AD respectively: shew that the triangles AXB, BYC are equal in area.

10. ABCD is a parallelogram, and P is any point within it; shew that the sum of the triangles PAB, PCD is equal to half the parallelogram.

EXERCISES ON THE AREA OF A TRIANGLE.

(*Numerical and Graphical.*)

1. The sides of a triangular field are 370 yds., 200 yds., and 190 yds. Draw a plan (scale 1″ to 100 yards). Draw and measure an altitude; hence calculate the approximate area of the field in square yards.

2. Two sides of a triangular enclosure are 124 metres and 144 metres respectively, and the included angle is observed to be 45°. Draw a plan (scale 1 cm. to 20 metres). Make any necessary measurement, and calculate the approximate area.

3. In a triangle ABC, given that the area = 6·6 sq. cm., and the base BC = 5·5 cm., find the altitude. Hence determine the locus of the vertex A.

If in addition to the above data, BA = 2·6 cm., construct the triangle; and measure CA.

4. In a triangle ABC, given *area* = 3·06 sq. in., and a = 3·0″. Find the altitude, and the locus of A. Given C = 68°, construct the triangle; and measure b.

5. ABC is a triangle in which BC, BA have constant lengths 6 cm. and 5 cm. If BC is fixed, and BA revolves about B, trace the changes in the area of the triangle as the angle B increases from 0° to 180°.

Answer this question by drawing a series of triangles, increasing B by increments of 30°. Find the area in each case and tabulate the results.

(*Theoretical.*)

6. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, shew that the triangles are equal *in area*. Can such triangles ever be *identically* equal?

7. Shew how to draw on the base of a given triangle an isosceles triangle of equal area.

8. If the middle points of the sides of a quadrilateral are joined in order, prove that the *parallelogram* so formed [see Ex. 7, p. 64], is half the quadrilateral.

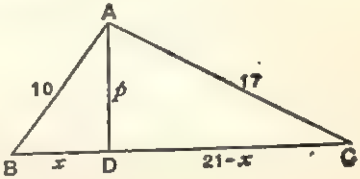
9. ABC is a triangle, and R, Q the middle points of the sides AB, AC; shew that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR.

10. Two triangles of equal area stand on the same base but on opposite sides of it: shew that the straight line joining their vertices is bisected by the base, or by the base produced.

[The method given below may be omitted from a first course. In any case it must be postponed till Theorem 29 has been read.]

The Area of a Triangle. *Given the three sides of a triangle, to calculate the area.*

EXAMPLE. *Find the area of a triangle whose sides measure* 21 m., 17 m., *and* 10 m.



Let ABC represent the given triangle.

Draw AD perp. to BC, and denote AD by p.

We shall first find the length of BD.

Let $BD = x$ metres; then $DC = 21 - x$ metres.

From the right-angled $\triangle ADB$, we have by Theorem 29

$$AD^2 = AB^2 - BD^2 = 10^2 - x^2.$$

And from the right-angled $\triangle ADC$,

$$AD^2 = AC^2 - DC^2 = 17^2 - (21-x)^2;$$

$$\therefore\ 10^2 - x^2 = 17^2 - (21-x)^2$$

or

$$100 - x^2 = 289 - 441 + 42x - x^2;$$

whence

$$x = 6.$$

Again,

$$AD^2 = AB^2 - BD^2;$$

or

$$p^2 = 10^2 - 6^2 = 64;$$

$$\therefore\ p = 8.$$

Now *Area of triangle* $= \frac{1}{2}$. *base* $\times$ *altitude*

$$= (\tfrac{1}{2} \times 21 \times 8) \text{ sq. m.} = 84 \text{ sq. m.}$$

EXERCISES.

Find by the above method the area of the triangles, whose sides are as follows:

1. 20 ft., 13 ft., 11 ft.
2. 15 yds., 14 yds., 13 yds.
3. 21 m., 20 m., 13 m.
4. 30 cm., 25 cm., 11 cm.
5. 37 ft., 30 ft., 13 ft.
6. 51 m., 37 m., 20 m.
7. If the given sides are a, b and c units in length, prove

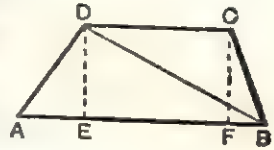
(i) $x = \dfrac{a^2 + c^2 - b^2}{2a}$; (ii) $p^2 = c^2 - \left\{\dfrac{a^2 + c^2 - b^2}{2a}\right\}^2$;

(iii) $\triangle = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$.

THE AREA OF QUADRILATERALS.

THEOREM 28.

To find the area of (i) *a trapezium.*
(ii) *any quadrilateral.*



(i) Let ABCD be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB.

Let the parallel sides AB, CD measure a and b units of length, and let the height CF contain h units.

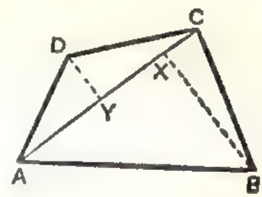
Then the area of ABCD $= \triangle ABD + \triangle DBC$

$$= \frac{1}{2}AB \,.\, DE + \frac{1}{2}DC \,.\, CF$$

$$= \frac{1}{2}ah + \frac{1}{2}bh = \frac{h}{2}(a+b).$$

That is,

the area of a trapezium $= \frac{1}{2}$ *height* $\times$ (*the sum of the parallel sides*).



(ii) Let ABCD be any quadrilateral. Draw a diagonal AC; and from B and D draw perpendiculars BX, DY to AC. These perpendiculars are called offsets.

If AC contains d units of length, and BX, DY p and q units respectively,

the area of the quad^l ABCD $= \triangle ABC + \triangle ADC$

$$= \frac{1}{2}AC \,.\, BX + \frac{1}{2}AC \,.\, DY$$

$$= \frac{1}{2}dp + \frac{1}{2}dq = \frac{1}{2}d(p+q).$$

That is to say,

the area of a quadrilateral $= \frac{1}{2}$ *diagonal* $\times$ (*sum of offsets*).

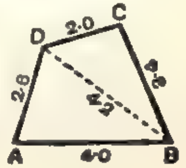
EXERCISES.

(Numerical and Graphical.)

1. Find the area of the trapezium in which the two parallel sides are 4·7" and 3·3", and the height 1·5".

2. In a quadrilateral ABCD, the diagonal AC=17 feet; and the offsets from it to B and D are 11 feet and 9 feet. Find the area.

3. In a plan ABCD of a quadrilateral enclosure, the diagonal AC measures 8·2 cm., and the offsets from it to B and D are 3·4 cm. and 2·6 cm. respectively. If 1 cm. in the plan represents 5 metres, find the area of the enclosure.



4. Draw a quadrilateral ABCD from the adjoining rough plan, the dimensions being given in inches.

Draw and measure the offsets to A and C from the diagonal BD; and hence calculate the area of the quadrilateral.

5. Draw a quadrilateral ABCD from the details given in the adjoining plan. The dimensions are to be in centimetres.

Make any necessary measurements of your figure, and calculate its area.

6. Draw a trapezium ABCD from the following data: AB and CD are the parallel sides. AB=4"; AD=BC=2"; the ∠A=the ∠B=60°.

Make any necessary measurements, and calculate the area.

7. Draw a trapezium ABCD in which AB and CD are the parallel sides; and AB=9 cm., CD=3 cm., and AD=BC=5 cm.

Make any necessary measurement, and calculate the area.

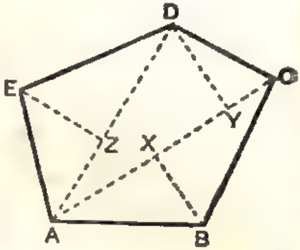
8. From the formula *area of quad^l* $=\frac{1}{2}$ *diag.* × (*sum of offsets*) shew that, if the diagonals are *at right angles*,

area $=\frac{1}{2}$ (*product of diagonals*).

9. Given the lengths of the diagonals of a quadrilateral, and the angle between them, prove that the area is the same wherever they intersect.

THE AREA OF ANY RECTILINEAL FIGURE.

1st METHOD. A rectilineal figure may be divided into triangles whose areas can be separately calculated from suitable measurements. The sum of these areas will be the area of the given figure.

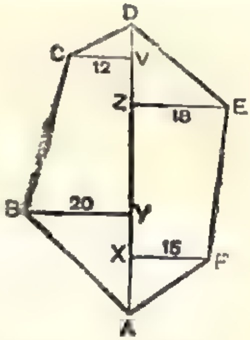


Example. The measurements required to find the area of the figure ABCDE are AC, AD, and the offsets BX, DY, EZ.

2nd METHOD. The area of a rectilineal figure is also found by taking a base-line (AD in the diagram below) and offsets from it. These divide the figure into *right-angled* triangles and *right-angled* trapeziums, whose areas may be found after measuring the offsets and the various sections of the base-line.

Example. Find the area of the enclosure ABCDEF from the plan and measurements tabulated below.

	Yards.	
	AD=56	
VC=12	AV=50	
	AZ=40	ZE=18
YB=20	AY=18	
	AX=10	XF=15



The measurements are made from A along the base line to the points from which the offsets spring.

Here

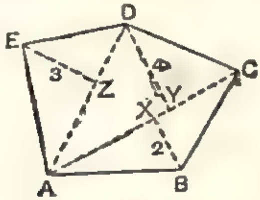
		Sq. yds.	Sq. yds.
$\triangle$ AXF	$=\frac{1}{2}$. AX × XF	$=\frac{1}{2}\times 10\times 15=$	75
$\triangle$ AYB	$=\frac{1}{2}$. AY × YB	$=\frac{1}{2}\times 18\times 20=$	180
$\triangle$ DZE	$=\frac{1}{2}$. DZ × ZE	$=\frac{1}{2}\times 16\times 18=$	144
$\triangle$ DVC	$=\frac{1}{2}$. DV × VC	$=\frac{1}{2}\times 6\times 12=$	36
trapm XFEZ	$=\frac{1}{2}$. XZ × (XF + ZE)	$=\frac{1}{2}\times 30\times 33=$	495
trapm YBCV	$=\frac{1}{2}$. YV × (YB + VC)	$=\frac{1}{2}\times 32\times 32=$	512

$\therefore$, by addition, the fig. ABCDEF = **1442** sq. yds.

EXERCISES.

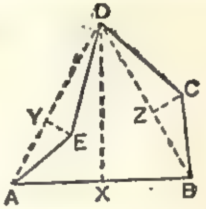
1. Calculate the areas of the figures (i) and (ii) from the plans and dimensions (in cms.) given below.

(i)



AC = 6 cm., AD = 5 cm.
Lengths of offsets figured in diagram.

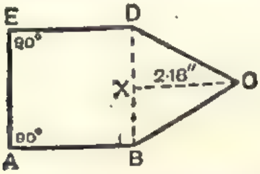
(ii)



AB = BD = DA = 6 cm.
EY = CZ = 1 cm.
DX = 5·2 cm.

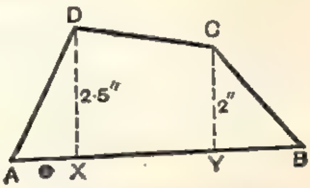
2. Draw full size the figures whose plans and dimensions are given below; and calculate the area in each case.

(i)



The fig. is equilateral: each side to be $2\frac{1}{2}$".

(ii)

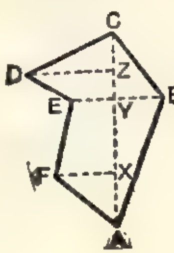


AX = $1\frac{1}{4}$", XY = $2\frac{3}{4}$", YB = $1\frac{3}{4}$".

3. Find the area of the figure ABCDEF from the following measurements and draw a plan in which 1 cm. represents 20 metres.

	Metres.	
	to C	
	180	
80 to D	150	
40 to E	120	50 to B
60 to F	50	
	From A	

The Plan.



EXERCISES ON QUADRILATERALS.

(Theoretical.)

1. ABCD is a rectangle, and PQRS the figure formed by joining in order the middle points of the sides.

Prove (i) that PQRS is a rhombus;

(ii) that the area of PQRS is half that of ABCD.

Hence shew that *the area of a rhombus is half the product of its diagonals.*

Is this true of any quadrilateral whose diagonals cut at right angles? Illustrate your answer by a diagram.

2. Prove that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals.

Hence shew how a parallelogram ABCD may be bisected by a straight line drawn

(i) through a given point P;

(ii) perpendicular to the side AB;

(iii) parallel to a given line QR.

3. In the trapezium ABCD, AB is parallel to DC; and X is the middle point of BC. Through X draw PQ parallel to AD to meet AB and DC produced at P and Q. Then prove

(i) trapezium ABCD = par^{m} APQD.

(ii) trapezium ABCD = twice the $\triangle$AXD.

(Graphical.)

4. The diagonals of a quadrilateral ABCD cut at right angles, and measure 3·0″ and 2·2″ respectively. Find the area.

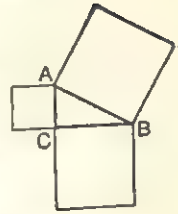
Shew by a figure that the area is the same wherever the diagonals cut, so long as they are at right angles.

5. In the parallelogram ABCD, AB = 8·0 cm., AD = 3·2 cm., and the perpendicular distance between AB and DC = 3·0 cm. Draw the parallelogram. Calculate the distance between AD and BC; and check your result by measurement.

6. One side of a parallelogram is 2·5″, and its diagonals are 3·4″ and 2·4″. Construct the parallelogram; and, after making any necessary measurement, calculate the area.

7. ABCD is a parallelogram on a fixed base AB and of constant area. Find the locus of the intersection of its diagonals.

EXERCISES LEADING TO THEOREM 29.



In the adjoining diagram, ABC is a triangle *right-angled* at C; and squares are drawn on the three sides. Let us compare the area of the square on the hypotenuse AB with the *sum* of the squares on the sides AC, CB which contain the right angle.

1. Draw the above diagram, making AC=3 cm., and BC=4 cm.;

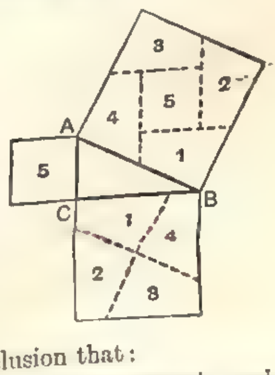
$$\left.\begin{array}{l}\text{Then the area of the square on AC}=3^2\text{, or } 9 \text{ sq. cm.}\\ \text{and .... the square on BC}=4^2\text{, or } 16 \text{ sq. cm.}\end{array}\right\}$$

$\therefore$ the sum of the squares on AC, BC= 25 sq. cm.

Now *measure* AB; hence calculate the area of the square on AB, and compare the result with the *sum* already obtained.

2. Repeat the process of the last exercise, making AC=1·0", and BC=2·4".

3. If $a=15$, $b=8$, $c=17$, shew arithmetically that $c^2=a^2+b^2$.
Now draw on squared paper a triangle ABC, whose sides a, b, and c are 15, 8, and 17 units of length; and *measure the angle* ACB.



4. Take any triangle ABC, right-angled at C; and draw squares on AC, CB, and on the hypotenuse AB.

Through the mid-point of the square on CB (*i.e.* the intersection of the diagonals) draw lines parallel and perpendicular to the hypotenuse, thus dividing the square into four *congruent* quadrilaterals. These, together with the square on AC, will be found exactly to fit into the square on AB, in the way indicated by corresponding numbers.

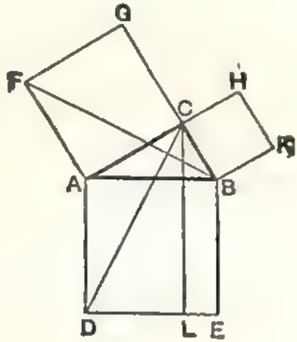
These experiments point to the conclusion that:

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

A formal proof of this theorem is given on the next page.

THEOREM 29. [Euclid I. 47.]

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled △, having the angle ACB a rt. ∠.

It is required to prove that the square on the hypotenuse AB = *the sum of the squares on* AC, CB.

On AB describe the sq. ADEB; and on AC, CB describe the sqq. ACGF, CBKH.

Through C draw CL par[l] to AD or BE.

Join CD, FB.

Proof. Because each of the ∠[s] ACB, ACG is a rt. ∠,
∴ BC and CG are in the same st. line.

Now the rt. ∠ BAD = the rt. ∠ FAC;
add to each the ∠ CAB:
then the whole ∠ CAD = the whole ∠ FAB.

Then in the △[s] CAD, FAB,

because { CA = FA,
AD = AB,
and the included ∠ CAD = the included ∠ FAB;

∴ the △ CAD = the △ FAB. *Theor.* 4.

Now the rect. AL is double of the $\triangle$CAD, being on the same base AD, and between the same parls AD, CL.

And the sq. GA is double of the $\triangle$FAB, being on the same base FA, and between the same parls FA, GB.

$\therefore$ the rect. AL = the sq. GA.

Similarly by joining CE, AK, it can be shewn that

the rect. BL = the sq. HB.

$\therefore$ the whole sq. AE = the sum of the sqq. GA, HB:

that is, the square on the hypotenuse AB = the sum of the squares on the two sides AC, CB.

Q.E.D.

Obs. This is known as the Theorem of Pythagoras. The result established may be stated as follows:

$$AB^2 = BC^2 + CA^2.$$

That is, if a and b denote the lengths of the sides containing the right angle; and if c denotes the hypotenuse,

$$c^2 = a^2 + b^2.$$

Hence $a^2 = c^2 - b^2$; and $b^2 = c^2 - a^2$.

NOTE 1. The following important results should be noticed. If CL and AB intersect in O, it has been shewn in the course of the proof that

the sq. GA = the rect. AL;
that is, AC^2 = the rect. contained by AB, AO. ....(i)

Also the sq. HB = the rect. BL;
that is, BC^2 = the rect. contained by BA, BO. ....(ii)

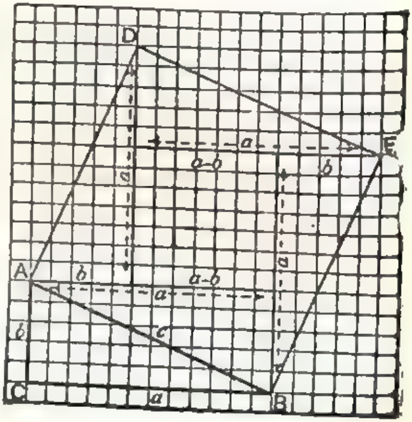
NOTE 2. It can be proved by superposition that *squares standing on equal sides are equal in area.*

Hence we conclude, conversely,

If two squares are equal in area they stand on equal sides.

EXPERIMENTAL PROOFS OF PYTHAGORAS'S THEOREM.

I. Here ABC is the given rt.-angled △; and ABED is the square on the hypotenuse AB.



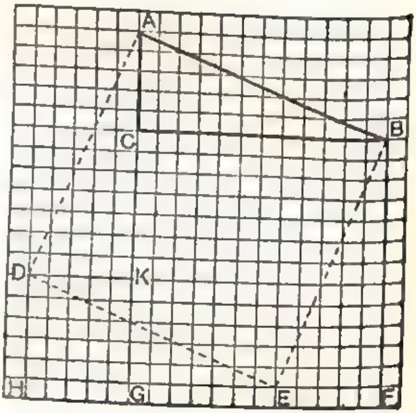
By drawing lines par[l] to the sides BC, CA, it is easily seen that the sq. BD is divided into 4 rt.-angled △s, each identically equal to ABC, together with a central square.

Hence

sq. on hypotenuse $c = 4$ rt. ∠d △s + the central square

$$= 4 \cdot \tfrac{1}{2}ab + (a-b)^2$$
$$= 2ab + a^2 - 2ab + b^2$$
$$= a^2 + b^2.$$

II. Here ABC is the given rt.-angled △, and the figs. CF, HK are the sqq. on CB, CA placed side by side.



FE is made equal to DH or CA; and the two sqq. CF, HK are cut along the lines BE, ED.

Then it will be found that the △DHE may be placed so as to fill up the space ACB; and the △BFE may be made to fill the space AKD.

Hence the two sqq. CF, HK may be fitted together so as to form the single fig. ABED, which will be found to be a perfect square, namely the square on the hypotenuse AB.

EXERCISES.

(Numerical and Graphical.)

1. Draw a triangle ABC, right-angled at C, having given:

(i) $a=3$ cm., $b=4$ cm.;

(ii) $a=2{\cdot}5$ cm., $b=6{\cdot}0$ cm.;

(iii) $a=1{\cdot}2''$, $b=3{\cdot}5''$.

In each case calculate the length of the hypotenuse c, and verify your result by measurement.

2. Draw a triangle ABC, right-angled at C, having given:

(i) $c=3{\cdot}4''$, $a=3{\cdot}0''$; [See Problem 10]

(ii) $c=5{\cdot}3$ cm., $b=4{\cdot}5$ cm.

In each case calculate the remaining side, and verify your result by measurement.

(The following examples are to be solved by calculation; but in each case a plan should be drawn on some suitable scale, and the calculated result verified by measurement.)

3. A ladder whose foot is 9 feet from the front of a house reaches to a window-sill 40 feet above the ground. What is the length of the ladder?

4. A ship sails 33 miles due South, and then 56 miles due West. How far is it then from its starting point?

5. Two ships are observed from a signal station to bear respectively N.E. 6·0 km. distant, and N.W. 1·1 km. distant. How far are they apart?

6. A ladder 65 feet long reaches to a point in the face of a house 63 feet above the ground. How far is the foot from the house?

7. B is due East of A, but at an unknown distance. C is due South of B, and distant 55 metres. AC is known to be 73 metres. Find AB.

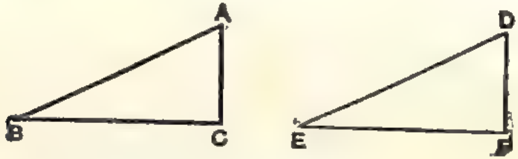
8. A man travels 27 miles due South; then 24 miles due West; finally 20 miles due North. How far is he from his starting point?

9. From A go West 25 metres, then North 60 metres, then East 80 metres, finally South 12 metres. How far are you then from A?

10. A ladder 50 feet long is placed so as to reach a window 48 feet high; and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street.

THEOREM 30. [Euclid I. 48.]

If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, then the angle contained by these two sides is a right angle.



Let ABC be a triangle in which
the sq. on AB = the sum of the sqq. on BC, CA.

It is required to prove that ACB *is a right angle.*

Make EF equal to BC.
Draw FD perp^r to EF, and make FD equal to CA.
Join ED.

Proof. Because EF = BC,
∴ the sq. on EF = the sq. on BC.
And because FD = CA,
∴ the sq. on FD = the sq. on CA.
Hence the sum of the sqq. on EF, FD = the sum of the sqq. on BC, CA.

But since EFD is a rt. ∠,
∴ the sum of the sqq. on EF, FD = the sq. on DE: *Theor.* 29.
And, by hypothesis, the sqq. on BC, CA = the sq. on AB.
∴ the sq. on DE = the sq. on AB.
∴ DE = AB.

Then in the $\triangle^s$ ACB, DFE,

because $\begin{cases} \text{AC} = \text{DF}, \\ \text{CB} = \text{FE}, \\ \text{and AB} = \text{DE}; \end{cases}$

∴ the ∠ ACB = the ∠ DFE. *Theor.* 7.
But, by construction, DFE is a right angle;
∴ the ∠ ACB is a right angle.

Q.E.D.

EXERCISES ON THEOREMS 29, 30.

(Theoretical.)

1. Shew that the square on the diagonal of a given square is double of the given square.

2. In the $\triangle ABC$, AD is drawn perpendicular to the base BC. If the side c is greater than b,

$$\text{shew that } c^2 - b^2 = BD^2 - DC^2.$$

3. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to BC, CA, AB respectively: shew that

$$AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2.$$

4. ABC is a triangle right-angled at A; and the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined. Prove that

$$BQ^2 + PC^2 = BC^2 + PQ^2.$$

5. In a right-angled triangle four times the sum of the squares on the medians drawn from the acute angles is equal to five times the square on the hypotenuse.

6. Describe a square equal to the sum of two given squares.

7. Describe a square equal to the difference between two given squares.

8. Divide a straight line into two parts so that the square on one part may be twice the square on the other.

9. Divide a straight line into two parts such that the sum of their squares shall be equal to a given square.

(Numerical and Graphical.)

10. Determine which of the following triangles are right-angled:

(i) $a = 14$ cm., $b = 48$ cm., $c = 50$ cm.;
(ii) $a = 40$ cm., $b = 10$ cm., $c = 41$ cm.;
(iii) $a = 20$ cm., $b = 99$ cm., $c = 101$ cm.

11. ABC is an isosceles triangle right-angled at C; deduce from Theorem 29 that

$$AB^2 = 2AC^2.$$

Illustrate this result graphically by drawing both diagonals of the square on AB, and one diagonal of the square on AC.

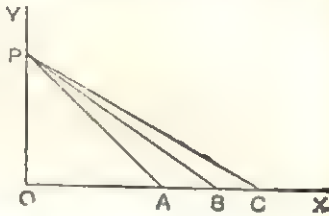
If AC = BC = 2″, find AB to the nearest hundredth of an inch, and verify your calculation by actual construction and measurement.

12. Draw a square on a diagonal of 6 cm. Calculate, and also measure, the length of a side. Find the area.

PROBLEM 16.

To draw squares whose areas shall be respectively twice, three-times, four-times, ..., that of a given square.

Hence find graphically approximate values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, ...



Take OX, OY at right angles to one another, and from them mark off OA, OP, each *one* unit of length. Join PA.

Then $PA^2 = OP^2 + OA^2 = 1 + 1 = 2.$

$\therefore\ PA = \sqrt{2}.$

From OX mark off OB equal to PA, and join PB;

then $PB^2 = OP^2 + OB^2 = 1 + 2 = 3.$

$\therefore\ PB = \sqrt{3}.$

From OX mark off OC equal to PB, and join PC;

then $PC^2 = OP^2 + OC^2 = 1 + 3 = 4.$

$\therefore\ PC = \sqrt{4}.$

The lengths of PA, PB, PC may now be found *by measurement*; and by continuing the process we may find $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, ...

EXERCISES ON THEOREMS 29, 30 (*Continued*).

13. Prove the following formula:

Diagonal of square = side $\times \sqrt{2}$.

Hence find to the nearest centimetre the diagonal of a square on a side of 50 metres.

Draw a plan (scale 1 cm. to 10 metres) and obtain the result as nearly as you can by measurement.

14. ABC is an equilateral triangle of which each side $= 2m$ units, and the perpendicular from any vertex to the opposite side $= p$.

Prove that $p = m\sqrt{3}$.

Test this result graphically, when each side = 8 cm.

15. If in a triangle $a=m^2-n^2$, $b=2mn$, $c=m^2+n^2$; prove algebraically that $c^2=a^2+b^2$.

Hence by giving various numerical values to m and n, find sets of numbers representing the sides of right-angled triangles.

16. In a triangle ABC, AD is drawn perpendicular to BC. Let p denote the length of AD.

(i) If $a=25$ cm., $p=12$ cm., BD$=9$ cm.; find b and c.

(ii) If $b=41''$, $c=50''$, BD$=30''$; find p and a.

And prove that $\sqrt{b^2-p^2}+\sqrt{c^2-p^2}=a$.

17. In the triangle ABC, AD is drawn perpendicular to BC. Prove that

$$c^2-\text{BD}^2=b^2-\text{CD}^2.$$

If $a=51$ cm., $b=20$ cm., $c=37$ cm.; find BD.

Thence find p, the length of AD, and the area of the triangle ABC.

18. Find by the method of the last example the areas of the triangles whose sides are as follows:

(i) $a=17''$, $b=10''$, $c=9''$. (ii) $a=25$ ft., $b=17$ ft., $c=12$ ft.

(iii) $a=41$ cm., $b=28$ cm., $c=15$ cm. (iv) $a=40$ yd., $b=37$ yd., $c=13$ yd.

19. A straight rod PQ slides between two straight rulers OX, OY placed at right angles to one another. In one position of the rod OP$=5{\cdot}6$ cm., and OQ$=3{\cdot}3$ cm. If in another position OP$=4{\cdot}0$ cm., find OQ graphically; and test the accuracy of your drawing by calculation.

20. ABC is a triangle right-angled at C, and p is the length of the perpendicular from C on AB. By expressing the area of the triangle in two ways, shew that

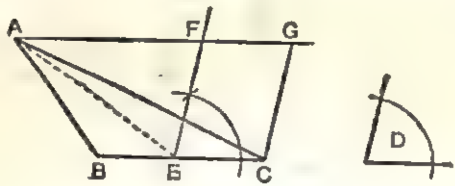
$$pc=ab.$$

Hence deduce
$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

PROBLEMS ON AREAS.

Problem 17.

To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.



Let ABC be the given triangle, and D the given angle.

It is required to describe a parallelogram equal to ABC, *and having one of its angles equal to* D.

Construction. Bisect BC at E.

At E in CE, make the ∠CEF equal to D;
through A draw AFG par^l to BC;
and through C draw CG par^l to EF.
Then FECG is the required par^m.

Proof. Join AE.

Now the $\triangle^s$ ABE, AEC are on equal bases BE, EC, and of the same altitude;

∴ the $\triangle$ ABE = the $\triangle$ AEC.
∴ the $\triangle$ ABC is double of the $\triangle$ AEC.

But FECG is a par^m by construction;
and it is double of the $\triangle$ AEC,
being on the same base EC, and between the same par^{ls} EC and AG.

∴ the par^m FECG = the $\triangle$ ABC;
and one of its angles, namely CEF, = the given ∠D.

EXERCISES.

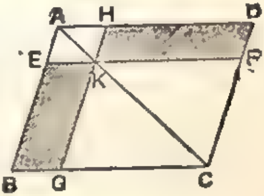
(Graphical.)

1. Draw a square on a side of 5 cm., and make a parallelogram of equal area on the same base, and having an angle of 45°.

Find (i) by calculation, (ii) by measurement the length of an oblique side of the parallelogram.

2. Draw any parallelogram ABCD in which AB=$2\frac{1}{2}$" and AD=2"; and on the base AB draw a rhombus of equal area.

DEFINITION In a parallelogram ABCD, if through any point K in the diagonal AC parallels EF, HG are drawn to the sides, then the figures EH, GF are called parallelograms about AC, and the figures EG, HF are said to be their complements.



3 *In the diagram of the preceding definition shew by Theorem 21 that the complements EG, HF are equal in area.*

Hence, given a parallelogram EG, and a straight line HK, deduce a construction for drawing on HK as one side a parallelogram equal and equiangular to the parallelogram EG.

4. Construct a rectangle equal in area to a given rectangle CDEF, and having one side equal to a given line AB.

If AB=6 cm., CD=8 cm., CF=3 cm., find by measurement the remaining side of the constructed rectangle.

5. Given a parallelogram ABCD, in which AB=2·4", AD=1·8", and the ∠A=55°. Construct an equiangular parallelogram of equal area, the greater side measuring 2·7". Measure the shorter side.

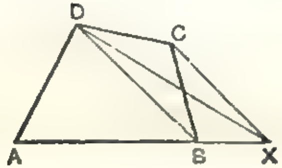
Repeat the process giving to A any other value; and compare your results. What conclusion do you draw?

6. Draw a rectangle on a side of 5 cm. equal in area to an equilateral triangle on a side of 6 cm.

Measure the remaining side of the rectangle, and calculate its approximate area.

Problem 18.

To draw a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

It is required to describe a triangle equal to ABCD *in area.*

Construction. Join DB.
Through C draw CX par[l] to DB, meeting AB produced in X.
Join DX.
Then DAX is the required triangle.

Proof. Now the △[s] XDB, CDB are on the same base DB and between the same par[ls] DB, CX;

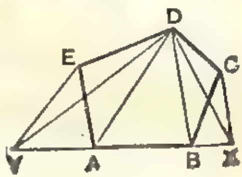
∴ the △ XDB = the △ CDB in area.

To each of these equals add the △ ADB;
then the △ DAX = the fig. ABCD.

Corollary. In the same way it is always possible to draw a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure; and thus step by step, any rectilineal figure may be reduced to a triangle of equal area.

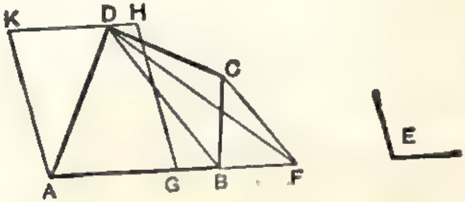
For example, in the adjoining diagram the five-sided fig. EDCBA is equal in area to the four-sided fig. EDXA.

The fig. EDXA may now be reduced to an equal △ DXY.



Problem 19.

To draw a parallelogram equal in area to a given rectilineal figure, and having an angle equal to a given angle.



Let ABCD be the given rectil. fig., and E the given angle.

It is required to draw a par^{m} equal to ABCD *and having an angle equal to* E.

Construction. Join DB.

Through C draw CF par^{l} to DB, and meeting AB produced in F.

Join DF.

Then the $\triangle$ DAF = the fig. ABCD. *Prob.* 18.

Draw the par^{m} AGHK equal to the $\triangle$ ADF, and having the $\angle$ KAG equal to the $\angle$ E. *Prob.* 17.

Then the par^{m} KG = the $\triangle$ ADF

= the fig. ABCD;

and it has the $\angle$ KAG equal to the $\angle$ E.

Note. If the given rectilineal figure has more than four sides, it must first be reduced, step by step, until it is replaced by an equivalent triangle.

EXERCISES.

(Reduction of a Rectilineal Figure to an equivalent Triangle.)

1. Draw a quadrilateral ABCD from the following data:

$AB = BC = 5{\cdot}5$ cm.; $CD = DA = 4{\cdot}5$ cm.; the $\angle A = 75°$.

Reduce the quadrilateral to a triangle of equal area. Measure the base and altitude of the triangle; and hence calculate the approximate area of the given figure.

2. Draw a quadrilateral ABCD having given:

$AB = 2{\cdot}8''$, $BC = 3{\cdot}2''$, $CD = 3{\cdot}3''$, $DA = 3{\cdot}6''$, and the diagonal $BD = 3{\cdot}0''$.

Construct an equivalent triangle; and hence find the approximate area of the quadrilateral.

3. On a base AB, 4 cm. in length, describe an equilateral pentagon (5 sides), having each of the angles at A and B 108°.

Reduce the figure to a triangle of equal area; and by measuring its base and altitude, calculate the approximate area of the pentagon.

4. A quadrilateral field ABCD has the following measurements:

$AB = 450$ metres, $BC = 380$ m., $CD = 330$ m., $AD = 390$ m., and the diagonal $AC = 660$ m.

Draw a plan (scale 1 cm. to 50 metres). Reduce your plan to an equivalent triangle, and measure its base and altitude. Hence estimate the area of the field.

(Problems. State your construction, and give a theoretical proof.)

5. Reduce a triangle ABC to a triangle of equal area having its base BD of given length. (D lies in BC, or BC produced.)

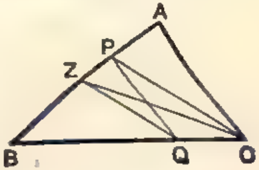
6. Construct a triangle equal in area to a given triangle, and having a given altitude.

7. ABC is a given triangle, and X a given point. Draw a triangle equal in area to ABC, having its vertex at X, and its base in the same straight line as BC.

8. Construct a triangle equal in area to the quadrilateral ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB.

9. Shew how a triangle may be divided into n equal parts by straight lines drawn through one of its angular points.

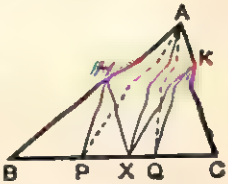
10. *Bisect a triangle by a straight line drawn through a given point in one of its sides.*



[Let ABC be the given Δ, and P the given point in the side AB.

Bisect AB at Z; and join CZ, CP.
Through Z draw ZQ parallel to CP.
Join PQ.
Then PQ bisects the Δ.]

11. *Trisect a triangle by straight lines drawn from a given point in one of its sides.*



[Let ABC be the given Δ, and X the given point in the side BC.

Trisect BC at the points P, Q. *Prob. 7.*
Join AX, and through P and Q draw PH and QK parallel to AX.
Join XH, XK.
These straight lines trisect the Δ; as may be shewn by joining AP, AQ.]

12. Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides.

13. *Bisect a quadrilateral by a straight line drawn through an angular point.*

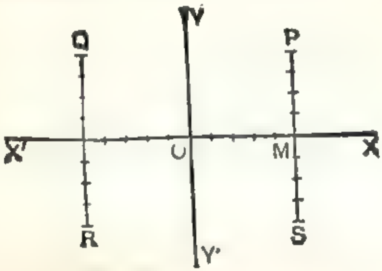
[Reduce the quadrilateral to a triangle of equal area, and join the vertex to the middle point of the base.]

14. Cut off from a given quadrilateral a third, a fourth, a fifth, or any part required, by a straight line drawn through a given angular point.

AXES OF REFERENCE. COORDINATES.

EXERCISES FOR SQUARED PAPER.

If we take two fixed straight lines XOX′, YOY′ cutting one another at right angles at O, the position of any point P with reference to these lines is known when we know its distances from each of them. Such lines are called **axes of reference**, XOX′ being known as the **axis of** x, and YOY′ as the **axis of** y Their point of intersection O is called the **origin.**



The lines XOX′, YOY′ are usually drawn horizontally and vertically.

In practice the distances of P from the axes are estimated thus:

From P, PM is drawn perpendicular to X′X; and OM and PM are measured.

OM is called the **abscissa** of the point P, and is denoted by x.
PM " **ordinate** " " " y.

The abscissa and ordinate taken together are called the **coordinates** of the point P, and are denoted by (x, y).

We may thus find the position of a point if its coordinates are known.

EXAMPLE. *Plot the point whose coordinates are* (5, 4).
Along OX mark off OM, 5 units in length.
At M draw MP perp. to OX, making MP 4 units in length.
Then P is the point whose coordinates are (5, 4).

The axes of reference divide the plane into four regions XOY, YOX′, X′OY′, Y′OX, known respectively as the first, second, third, and fourth **quadrants.**

It is clear that in each quadrant there is a point whose distances from the axes are equal to those of P in the above diagram, namely, 5 units and 4 units.

The coordinates of these points are distinguished by the use of the *positive* and *negative* signs, according to the following system:

Abscissæ measured along the x-axis to the right of the origin are positive, those measured to the left of the origin are negative. Ordinates which lie above the x-axis (that is, in the first and second quadrants) are positive; those which lie below the x-axis (that is, in the third and fourth quadrants) are negative.

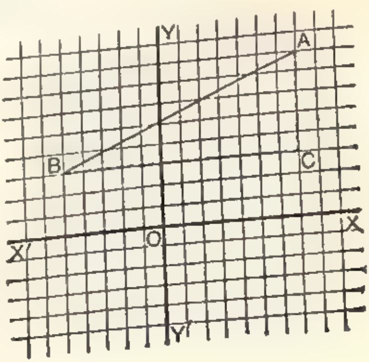
Thus the coordinates of the points Q, R, S are

$$(-5, 4),\ (-5, -4),\ \text{and}\ (5, -4)\ \text{respectively.}$$

NOTE. The coordinates of the origin are (0, 0).

In practice it is convenient to use squared paper. Two intersecting lines should be chosen as axes, and slightly thickened to aid the eye, then one or more of the length-divisions may be taken as the linear unit. The paper used in the following examples is ruled to tenths of an inch.

EXAMPLE 1. *The coordinates of the points* A *and* B *are* (7, 8) *and* (−5, 3) : *plot the points and find the distance between them.*



After plotting the points as in the diagram, we may find AB approximately by direct measurement.

Or we may proceed thus:

Draw through B a line par[l] to XX′ to meet the ordinate of A at C. Then ACB is a rt.-angled △ in which BC = 12, and AC = 5.

$$\text{Now } AB^2 = BC^2 + AC^2$$
$$= 12^2 + 5^2$$
$$= 144 + 25$$
$$= 169.$$

$$\therefore AB = 13.$$

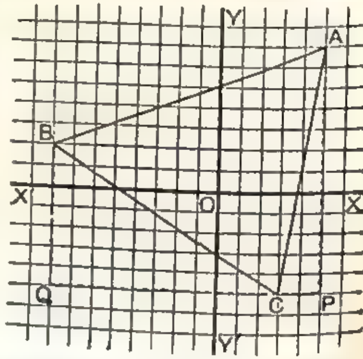
EXAMPLE 2. *The coordinates of* A, B, *and* C *are* (5, 7), (−8, 2), *and* (3, −5); *plot these points and find the area of the triangle of which the... are the vertices.*

Having plotted the points as in the diagram, we may measure AB, and draw and measure the perp. from C on AB. Hence the approximate area may be calculated.

Or we may proceed thus:

Through A and B draw AP, BQ par[l] to YY′.

Through C draw PQ par[l] to XX′.



Then the △ABC = the trap[m] APQB − the two rt.-angled △[s] APC, BQC

$= \frac{1}{2}PQ(AP + BQ) - \frac{1}{2}.AP.PC - \frac{1}{2}.BQ.QC$

$= \frac{1}{2} \times 13 \times 19 \quad - \frac{1}{2} \times 12 \times 2 \; - \frac{1}{2} \times 7 \times 11$

$= 73$ units of area.

EXERCISES.

1. Plot the following sets of points:

(i) (6, 4), (−6, 4), (−6, −4), (6, −4);
(ii) (8, 0), (0, 8), (−8, 0), (0, −8);
(iii) (12, 5), (5, 12), (−12, 5), (−5, 12).

2. Plot the following points, and shew experimentally that each set lie in one straight line.

(i) (9, 7), (0, 0), (−9, −7); (ii) (−9, 7), (0, 0), (9, −7)

Explain these results theoretically.

3. Plot the following pairs of points; join the ints in each case, and measure the coordinates of the mid-point of th ining line.

(i) (4, 3), (12, 7); (ii) (5, 4), (1

Shew *why* in each case the coordinates of the mid-point are respectively *half the sum of the abscissæ* and *half the sum of the ordinates* of the given points.

4. Plot the following pairs of points; and find the coordinates of the mid-point of their joining lines.

(i) (0, 0), (8, 10); (ii) (8, 0), (0, 10);
(iii) (0, 0), (−8, −10); (iv) (−8, 0), (0, −10).

5. Find the coordinates of the points of trisection of the line joining (0, 0) to (18, 15).

6. Plot the two following series of points:

(i) (5, 0), (5, 2), (5, 5), (5, −1), (5, −4);

(ii) (−4, 8), (−1, 8), (0, 8), (3, 8), (6, 8).

Shew that they lie on two lines respectively parallel to the axis of y, and the axis of x. Find the coordinates of the point in which they intersect.

7. Plot the following points, and calculate their distances from the origin.

(i) (15, 8); (ii) (−15, −8); (iii) (2·4", ·7"); (iv) (−·7", 2·4").

Check your results by measurement.

8. Plot the following pairs of points, and in each case calculate the distance between them.

(i) (4, 0), (0, 3); (ii) (9, 8), (5, 5);

(iii) (15, 0), (0, 8); (iv) (10, 4), (−5, 12);

(v) (20, 12), (−15, 0); (vi) (20, 9), (−15, −3).

Verify your calculation by measurement.

9. Shew that the points (−3, 2), (3, 10), (7, 2) are the angular points of an isosceles triangle. Calculate and measure the lengths of the equal sides.

10. Plot the eight points (0, 5), (3, 4), (5, 0), (4, −3), (−5, 0), (0, −5), (−4, 3), (−4, −3), and shew that they all lie on a circle whose centre is the origin.

11. Explain by a diagram why the distances between the following pairs of points are all equal.

(i) (a, 0), (0, b); (ii) (b, 0), (0, a); (iii) (0, 0), (a, b).

12. Draw the straight lines joining

(i) (a, 0) and (0, a); (ii) (0, 0) and (a, a);

and prove that these lines bisect each other at right angles.

13. Shew that (0, 4), (12, 9), (12, −4) are the vertices of an isosceles triangle whose base is bisected by the axis of x.

14. Three vertices of a rectangle are (14, 0), (14, 10), and (0, 10); find the coordinates of the fourth vertex, and of the intersection of the diagonals.

15. Prove that the four points (0, 0), (13, 0), (18, 12), (5, 12) are the angular points of a rhombus. Find the length of each side, and the coordinates of the intersection of the diagonals.

16. Plot the locus of a point which moves so that its distances from the points (0, 0) and (4, −4) are always equal to one another. Where does the locus cut the axes?

17. Shew that the following groups of points are the vertices of rectangles. Draw the figures, and calculate their areas.

(i) (4, 3), (17, 3), (17, 12), (4, 12);
(ii) (3, 2), (3, 15), (−6, 15), (−6, 2);
(iii) (5, 1), (−8, 1), (−8, −8), (5, −8).

18. Join in order the points (1″, 0), (0, 1″), (−1″, 0), (0, −1″). Of what kind is the quadrilateral so formed? Find its area.

If a second figure is formed by joining the middle points of the first, find its area.

19. Plot the triangles given by the following sets of points; and find their areas.

(i) (10, 10), (4, 0,), (18, 0); (ii) (10, −10), (4, 0), (18, 0);
(iii) (−10, 10), (−4, 0), (−18, 0); (iv) (−10, −10), (−4, 0), (−18, 0).

20. Draw the triangles given by the points

(i) (0, 0), (5, 3), (6, 0); (ii) (0, 0), (3, 0), (0, 6).

Find their areas; and measure the angles of the first triangle.

21. Plot the triangles given by the following sets of points. Shew that in each case one side is parallel to one of the axes. Hence find the area.

(i) (0, 0), (12, 10), (12, −6); (ii) (0, 0), (5, 8), (−15, 8);
(iii) (0, 0), (−12, 12,), (−12, −8); (iv) (0, 0), (−6, −8), (20, −8).

22. In the following triangles shew that two sides of each are parallel to the axes. Find their areas.

(i) (5, 5), (15, 5), (15, 15); (ii) (8, 3), (8, 18), (0, 18);
(iii) (4, 8), (−16, −4), (4, −4); (iv) (1, 15), (−11, 15), (1, −7).

23. Shew that (−5, 5), (7, 10), (10, 6), (−2, 1) are the angular points of a parallelogram. Find its sides and area.

24. Shew that each of the following sets of points gives a trapezium. Find the area of each.

(i) (3, 0), (3, 3), (9, 0), (9, 6); (ii) (0, 3), (−5, 3), (−2, −3), (0, −3);
(iii) (8, 4), (4, 4), (11, −1), (3, −1); (iv) (0, 0), (−1, 5), (−4, 5), (−8, 0).

25. Find the area of the triangles given by the following points:

(i) (5, 5), (20, 10), (12, 14); (ii) (7, 6), (−10, 4), (−4, −3);
(iii) (0, −6), (0, −3), (14, 5); (iv) (6, 4), (−7, −6), (−2, −15).

26. Shew that (−5, 0), (7, 5), (19, 0), (7, −5) are the angular points of a rhombus. Find its sides and its area.

27. Join the points $(0, -5)$, $(12, 0)$, $(4, 6)$, $(-8, -3)$, in the order given. Calculate the lengths of the first three sides and measure the fourth. Find the areas of the portions of the figure lying in the first and fourth quadrants.

28. The coordinates of four points A, B, C, D are respectively

$$(-4, -4), \ (-10, 4), \ (-10, 13), \ (5, 5).$$

Calculate the lengths of AB, BC, CD, and measure AD. Also calculate the area of ABCD by considering it as the difference of two triangles.

29. Draw the figure whose angular points are given by

$$(0, -3), \ (8, 3), \ (-4, 8), \ (-4, 3), \ (0, 0).$$

Find the lengths of its sides, taking the points in the above order. Also divide it into three right-angled triangles, and hence find its area.

30. A plan of a triangular field ABC is drawn on squared paper (scale $1'' = 100$ yds.). On the plan the coordinates of A, B, C are $(-1'', -3'')$, $(3'', 4'')$, $(-5'', -2'')$ respectively. Find the area of the field, the length of the side represented by BC, and the distance from this side of the opposite corner of the field.

31. Shew that the points $(6, 0)$, $(20, 6)$, $(14, 20)$, $(0, 14)$ are the vertices of a square. Measure a side and hence find the approximate area. Calculate the area exactly (i) by drawing a circumscribing square through its vertices; (ii) by subdividing the given square as in the first figure on page 120.

MISCELLANEOUS EXERCISES.

1. AB and AC are unequal sides of a triangle ABC; AX is the median through A, AP bisects the angle BAC, and AD is the perpendicular from A to BC. Prove that AP is intermediate in position and magnitude to AX and AD.

2. In a triangle if a perpendicular is drawn from one extremity of the base to the bisector of the vertical angle, (i) it will make with either of the sides containing the vertical angle an angle equal to half the sum of the angles at the base; (ii) it will make with the base an angle equal to half the difference of the angles at the base.

3. In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.

4. Construct a right-angled triangle having given the hypotenuse and the difference of the other sides.

5. Construct a triangle, having given the base, the difference of the angles at the base, and (i) the difference, (ii) the sum of the remaining sides.

6. Construct an isosceles triangle, having given the base and the sum of one of the equal sides and the perpendicular from the vertex to the base.

7. Shew how to divide a given straight line so that the square on one part may be double of the square on the other.

8. ABCD is a parallelogram, and O is any point without the angle BAD or its opposite vertical angle; shew that the triangle OAC is equal to the sum of the triangles OAD, OAB.

If O is within the angle BAD or its opposite vertical angle, shew that the triangle OAC is equal to the difference of the triangles OAD, OAB.

9. The area of a quadrilateral is equal to the area of a triangle having two of its sides equal to the diagonals of the given figure, and the included angle equal to either of the angles between the diagonals.

10. Find the locus of the intersection of the medians of triangles described on a given base and of given area.

11. On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line.

12. ABCD is a parallelogram made of rods connected by hinges. If AB is fixed, find the locus of the middle point of CD.

PART III.

THE CIRCLE.

Definitions and First Principles.

1. **A circle** is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same.

The fixed point is called the **centre**, and the bounding line is called the **circumference**.

Note. According to this definition the term circle strictly applies to the *figure* contained by the circumference; it is often used however for the circumference itself when no confusion is likely to arise.

2. **A radius** of a circle is a straight line drawn from the centre to the circumference. It follows that all radii of a circle are equal.

3. **A diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

4. **A semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

It will be proved on page 142 that a diameter divides a circle into two identically equal parts.

5. Circles that have the same centre are said to be **concentric**.

From these definitions we draw the following inferences:

(i) A circle is a *closed* curve; so that if the circumference is crossed by a straight line, this line if produced will cross the circumference at a second point.

(ii) The distance of a point from the centre of a circle is greater or less than the radius according as the point is without or within the circumference.

(iii) A point is outside or inside a circle according as its distance from the centre is greater or less than the radius.

(iv) Circles of equal radii are identically equal. For by superposition of one centre on the other the circumferences must coincide at every point.

(v) Concentric circles of unequal radii cannot intersect, for the distance from the centre of every point on the smaller circle is less than the radius of the larger.

(vi) If the circumferences of two circles have a common point they cannot have the same centre, unless they coincide altogether.

6. An arc of a circle is any part of the circumference.

7. A chord of a circle is a straight line joining any two points on the circumference.

NOTE. From these definitions it may be seen that a chord of a circle, which does not pass through the centre, divides the circumference into two unequal arcs; of these, the greater is called the major arc, and the less the minor arc. Thus the major arc is *greater*, and the minor arc *less* than the semi-circumference.

The major and minor arcs, into which a circumference is divided by a chord, are said to be conjugate to one another.

SYMMETRY.

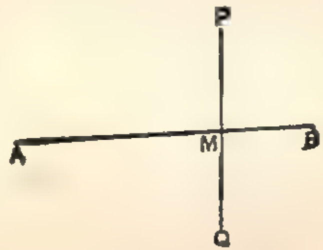
Some elementary properties of circles are easily proved by considerations of symmetry. For convenience the definition given on page 21 is here repeated.

DEFINITION 1. A figure is said to be **symmetrical about a line** when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence.

The straight line is called an **axis of symmetry.**

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis.

DEFINITION 2. Let AB be a straight line and P a point outside it.



From P draw PM perp. to AB, and produce it to Q, making MQ equal to PM.

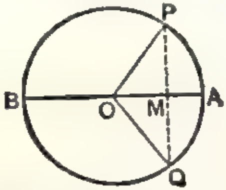
Then if the figure is folded about AB, the point P may be made to coincide with Q, for the $\angle$AMP = the $\angle$AMQ, and MP = MQ.

The points P and Q are said to be **symmetrically opposite** with regard to the axis AB, and each point is said to be the **image** of the other in the axis.

NOTE. A point and its image are equidistant from every point on the axis. See Prob. 14, page, 91.

Some Symmetrical Properties of Circles.

I. *A circle is symmetrical about any diameter.*



Let APBQ be a circle of which O is the centre, and AB any diameter.

It is required to prove that the circle is symmetrical about AB.

Proof. Let OP and OQ be two radii making any equal $\angle^s$ AOP, AOQ on opposite sides of OA.

Then if the figure is folded about AB, OP may be made to fall along OQ, since the $\angle$ AOP = the $\angle$ AOQ.

And thus P will coincide with Q, since OP = OQ.

Thus every point in the arc APB must coincide with some point in the arc AQB; that is, the two parts of the circumference on each side of AB can be made to coincide.

$\therefore$ the circle is symmetrical about the diameter AB.

Corollary. If PQ is drawn cutting AB at M, then on folding the figure about AB, since P falls on Q, MP will coincide with MQ,

$\therefore$ MP = MQ;

and the $\angle$ OMP will coincide with the $\angle$ OMQ,

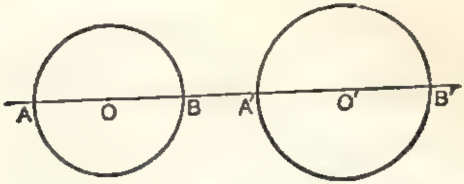
$\therefore$ these angles, being adjacent, are rt. $\angle^s$;

$\therefore$ the points P and Q are symmetrically opposite with regard to AB.

Hence, conversely, *if a circle passes through a given point* P, *it also passes through the symmetrically opposite point with regard to any diameter.*

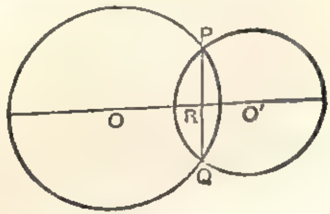
Definition. The straight line passing through the centres of two circles is called the line of centres.

II. *Two circles are divided symmetrically by their line of centres.*



Let O, O′ be the centres of two circles, and let the st. line through O, O′ cut the Oces at A, B and A′, B′. Then AB and A′B′ are diameters and therefore axes of symmetry of their respective circles. That is, the line of centres divides each circle symmetrically.

III. *If two circles cut at one point, they must also cut at a second point; and the common chord is bisected at right angles by the line of centres.*



Let the circles whose centres are O, O′ cut at the point P.

Draw PR perp. to OO′, and produce it to Q, so that RQ = RP.

Then P and Q are symmetrically opposite points with regard to the line of centres OO′;

∴ since P is on the Oce of both circles, it follows that Q is also on the Oce of both. [I. Cor.]

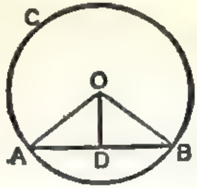
And, by construction, the common chord PQ is bisected at right angles by OO′.

ON CHORDS.

THEOREM 31. [Euclid III. 3.]

If a straight line drawn from the centre of a circle bisects a chord which does not pass through the centre, it cuts the chord at right angles.

Conversely, if it cuts the chord at right angles, it bisects it.



Let ABC be a circle whose centre is O; and let OD bisect a chord AB which does not pass through the centre.

It is required to prove that OD *is perp. to* AB.

Join OA, OB.

Proof. Then in the $\triangle^s$ ADO, BDO,

because $\begin{cases} \text{AD} = \text{BD, by hypothesis,} \\ \text{OD is common,} \\ \text{and OA} = \text{OB, being radii of the circle;} \end{cases}$

$\therefore$ the $\angle$ ADO = the $\angle$ BDO; *Theor.* 7.

and these are adjacent angles,

$\therefore$ OD is perp. to AB. Q.E.D.

Conversely. Let OD be perp. to the chord AB.

It is required to prove that OD *bisects* AB.

Proof. In the $\triangle^s$ ODA, ODB,

because $\begin{cases} \text{the } \angle^s \text{ ODA, ODB are } \textit{right angles,} \\ \text{the hypotenuse OA} = \text{the hypotenuse OB,} \\ \text{and OD is common;} \end{cases}$

$\therefore$ DA = DB; *Theor.* 18.

that is, OD bisects AB at D. Q.E.D.

Corollary 1. *The straight line which bisects a chord at right angles passes through the centre.*

Corollary 2. *A straight line cannot meet a circle at more than two points.*

For suppose a st. line meets a circle whose centre is O at the points A and B.

Draw OC perp. to AB.

Then AC = CB.

Now if the circle were to cut AB in a third point D, AC would also be equal to CD, which is impossible.

Corollary 3. *A chord of a circle lies wholly within it.*

EXERCISES.

(Numerical and Graphical.)

1. In the figure of Theorem 31, if AB = 8 cm., and OD = 3 cm., find OB. Draw the figure, and verify your result by measurement.

2. Calculate the length of a chord which stands at a distance 5″ from the centre of a circle whose radius is 13″.

3. In a circle of 1″ radius draw two chords 1·6″ and 1·2″ in length. Calculate and measure the distance of each from the centre.

4. Draw a circle whose diameter is 8·0 cm. and place in it a chord 6·0 cm. in length. Calculate to the nearest millimetre the distance of the chord from the centre; and verify your result by measurement.

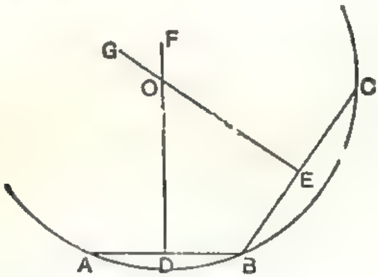
5. Find the distance from the centre of a chord 5 ft. 10 in. in length in a circle whose diameter is 2 yds. 2 in. Verify the result graphically by drawing a figure in which 1 cm. represents 10″.

6. AB is a chord 2·4″ long in a circle whose centre is O and whose radius is 1·3″; find the area of the triangle OAB in square inches.

7. Two points P and Q are 3″ apart. Draw a circle with radius 1·7″ to pass through P and Q. Calculate the distance of its centre from the chord PQ, and verify by measurement.

Theorem 32.

One circle, and only one, can pass through any three points not in the same straight line.



Let A, B, C be three points not in the same straight line.

It is required to prove that one circle, and only one, can pass through A, B, *and* C.

Join AB, BC,

Let AB and BC be bisected at right angles by the lines DF, EG.

Then since AB and BC are not in the same st. line, DF and EG are not par[l].

Let DF and EG meet in O.

Proof. Because DF bisects AB at right angles,

∴ every point on DF is equidistant from A and B. *Prob.* 14.

Similarly every point on EG is equidistant from B and C.

∴ O, the only point common to DF and EG, is equidistant from A, B, and C;

and there is no other point equidistant from A, B, and C.

∴ a circle having its centre at O and radius OA will pass through B and C; and this is the only circle which will pass through the three given points. Q.E.D.

COROLLARY 1. *The size and position of a circle are fully determined if it is known to pass through three given points;* for then the position of the centre and length of the radius can be found.

COROLLARY 2. *Two circles cannot cut one another in more than two points without coinciding entirely;* for if they cut at three points they would have the same centre and radius.

HYPOTHETICAL CONSTRUCTION. *From Theorem 32 it appears that we may suppose a circle to be drawn through any three points not in the same straight line.*

For example, a circle can be assumed to pass through the vertices of any triangle.

DEFINITIO . The circle which passes through the vertices of a triangle is called its circum-circle, and is said to be circumscribed about the triangle. The centre of the circle is called the circum-centre of the triangle, and the radius is called the circum-radius.

EXERCISES ON THEOREMS 31 AND 32.

(Theoretical.)

1. The par of a straight line intercepted between the circumferences of two concentric circles are equal.

2. Two circles, whose centres are at A and B, intersect at C, D; and M is the middle point of the common chord. Shew that AM and BM are in the same straight line.

Hence prove that the line of centres bisects the common chord at right angles.

3. AB, AC are two equal chords of a circle; shew that the straight line which bisects the angle BAC passes through the centre.

4. *Find th locus of the centres of all circles which pass through two given points.*

5. *Describe a circle that shall pass through two given points and have its centre in a given straight line.*

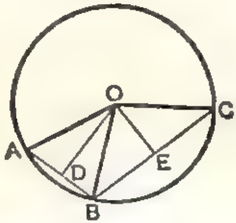
When is this impossible?

6. *Describe a circle of given radius to pass through two given points.*

When is this impossible?

* THEOREM 33. [Euclid III. 9.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and O a point within it from which more than two equal st. lines are drawn to the $\odot^{ce}$, namely OA, OB, OC.

It is required to prove that O *is the centre of the circle* ABC.

Join AB, BC.

Let D and E be the middle points of AB and BC respectively

Join OD, OE.

Proof. In the $\triangle^s$ ODA, ODB,

because $\left\{\begin{array}{l} \text{DA} = \text{DB}, \\ \text{DO is common,} \\ \text{and OA} = \text{OB, by hypothesis,} \end{array}\right.$

$\therefore$ the $\angle$ ODA = the $\angle$ ODB; *Theor.* 7

$\therefore$ these angles, being adjacent, are rt. $\angle^s$.

Hence DO bisects the chord AB at right angles, and therefore passes through the centre. *Theor.* 31. *Cor.* 1.

Similarly it may be shewn that EO passes through the centre.

$\therefore$ O, which is the only point common to DO and EO, must be the centre. Q.E.D.

EXERCISES ON CHORDS.

(*Numerical and Graphical.*)

1. AB and BC are lines at right angles, and their lengths are 1·6" and 3·0" respectively. Draw the circle through the points A, B, and C; find the length of its radius, and verify your result by measurement.

2. Draw a circle in which a chord 6 cm. in length stands at a distance of 3 cm. from the centre.

Calculate (to the nearest millimetre) the length of the radius, and verify your result by measurement.

3. Draw a circle on a diameter of 8 cm., and place in it a chord equal to the radius.

Calculate (to the nearest millimetre) the distance of the chord from the centre, and verify by measurement.

4. Two circles, whose radii are respectively 26 inches and 25 inches, intersect at two points which are 4 feet apart. Find the distances between their centres.

Draw the figure (scale 1 cm. to 10"), and verify your result by measurement.

5. Two parallel chords of a circle whose diameter is 13" are respectively 5" and 12" in length: shew that the distance between them is either 8·5" or 3·5".

6. Two parallel chords of a circle on the same side of the centre are 6 cm. and 8 cm. in length respectively, and the perpendicular distance between them is 1 cm. Calculate and measure the radius.

7. Shew on squared paper that if a circle has its centre at *any* point on the x-axis and passes through the point (6, 5), it also passes through the point (6, −5). [See page 132.]

(*Theoretical.*)

8. The line joining the middle points of two parallel chords of a circle passes through the centre.

9. *Find the locus of the middle points of parallel chords in a circle.*

10. Two intersecting chords of a circle cannot bisect each other unless each is a diameter.

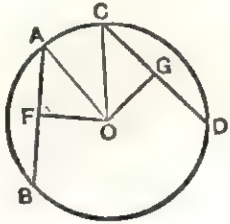
11. If a parallelogram can be inscribed in a circle, the point of intersection of its diagonals must be at the centre of the circle.

12. Shew that rectangles are the only parallelograms that can be inscribed in a circle.

THEOREM 34. [Euclid III. 14.]

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

First. Let AB = CD.

It is required to prove that AB *and* CD *are equidistant from* O.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

∴ OF bisects AB; *Theor.* 31.

∴ AF is half of AB.

Similarly CG is half of CD.

But, by hypothesis, AB = CD,

∴ AF = CG.

Now in the △ˢ OFA, OGC,

because $\left\{\begin{array}{l}\text{the } \angle^{\text{s}} \text{ OFA, OGC are } \textit{right angles}, \\ \text{the hypotenuse OA = the hypotenuse OC}, \\ \text{and AF = CG};\end{array}\right.$

∴ the triangles are equal in all respects; *Theor.* 18.

so that OF = OG;

that is, AB and CD are equidistant from O.

Q.E.D.

Conversely. Let $OF = OG$.

It is required to prove that $AB = CD$.

Proof. As before it may be shewn that AF is half of AB, and CG half of CD.

Then in the $\triangle^s$ OFA, OGC,

$$\text{because} \begin{cases} \text{the } \angle^s \text{ OFA, OGC are } \textit{right angles}, \\ \text{the hypotenuse OA} = \text{the hypotenuse OC}, \\ \text{and OF} = \text{OG}; \end{cases}$$

$\therefore$ $AF = CG$; *Theor.* 18.

$\therefore$ the doubles of these are equal;

that is, $AB = CD$.

Q.E.D.

EXERCISES.

(*Theoretical.*)

1. *Find the locus of the middle points of equal chords of a circle.*

2. If two chords of a circle cut one another, and make equal angles with the straight line which joins their point of intersection to the centre, they are equal.

3. If two equal chords of a circle intersect, shew that the segments of the one are equal respectively to the segments of the other.

4. In a given circle draw a chord which shall be equal to one given straight line (not greater than the diameter) and parallel to another.

5. PQ is a fixed chord in a circle, and AB is any diameter: shew that the sum or difference of the perpendiculars let fall from A and B on PQ is constant, that is, the same for all positions of AB.

[See Ex. 9, p. 65.]

(*Graphical.*)

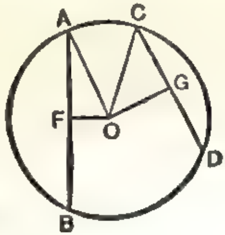
6. In a circle of radius 4·1 cm. any number of chords are drawn each 1·8 cm. in length. Shew that the middle points of these chords all lie on a circle. Calculate and measure the length of its radius, and draw the circle.

7. The centres of two circles are 4" apart, their common chord is 2·4" in length, and the radius of the larger circle is 3·7". Give a construction for finding the points of intersection of the two circles, and find the radius of the smaller circle.

THEOREM 35. [Euclid III. 15.]

Of any two chords of a circle, that which is nearer to the centre is greater than one more remote.

Conversely, the greater of two chords is nearer to the centre than the less.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O

It is required to prove that

(i) *if* OF *is less than* OG, *then* AB *is greater than* CD;

(ii) *if* AB *is greater than* CD, *then* OF *is less than* OG.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

∴ OF bisects AB;

∴ AF is half of AB.

Similarly CG is half of CD.

Now OA = OC,

∴ the sq. on OA = the sq. on OC

But since the ∠OFA is a rt. angle,

∴ the sq. on OA = the sqq. on OF, FA.

Similarly the sq. on OC = the sqq. on OG, GC.

∴ the sqq. on OF, FA = the sqq. on OG, GC.

(i) Hence if OF is given less than OG;
the sq. on OF is less than the sq. on OG.
∴ the sq. on FA is greater than the sq. on GC;
∴ FA is greater than GC:
∴ AB is greater than CD.

(ii) But if AB is given greater than CD,
that is, if FA is greater than GC;
then the sq. on FA is greater than the sq. on GC.
∴ the sq. on OF is less than the sq. on OG;
∴ OF is less than OG. Q.E.D.

COROLLARY. *The greatest chord in a circle is a diameter.*

EXERCISES.

(Miscellaneous.)

1. Through a given point within a circle draw the least possible chord.

2. Draw a triangle ABC in which $a=3 \cdot 5''$, $b=1 \cdot 2''$, $c=3 \cdot 7''$. Through the ends of the side a draw a circle with its centre on the side c. Calculate and measure the radius.

3. Draw the circum circle of a triangle whose sides are $2 \cdot 6''$, $2 \cdot 8''$, and $3 \cdot 0''$. Measure its radius

4. AB is a fixed chord of a circle, and XY any other chord having its middle point Z on AB; what is the greatest, and what the least length that XY may have?

Shew that XY increases, as Z approaches the middle point of AB.

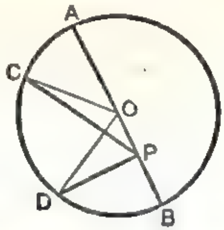
5. Shew on squared paper that a circle whose centre is at the origin, and whose radius is $3 \cdot 0''$, passes through the points $(2 \cdot 4'', 1 \cdot 8'')$, $(1 \cdot 8'', 2 \cdot 4'')$.

Find (i) the length of the chord joining these points, (ii) the co-ordinates of its middle point, (iii) its perpendicular distance from the origin.

*Theorem 36. [Euclid III. 7.]

If from any internal point, not the centre, straight lines are drawn to the circumference of a circle, then the greatest is that which passes through the centre, and the least is the remaining part of that diameter.

And of any other two such lines the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from P any internal point, which is not the centre, let PA, PB, PC, PD be drawn to the O^{ce}, so that PA passes through the centre O, and PB is the remaining part of that diameter. Also let the ∠POC at the centre subtended by PC be greater than the ∠POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*

(ii) PB *is the least,*

(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the △POC, the sides PO, OC are together greater than PC. *Theor.* 11.

But OC = OA, being radii;

∴ PO, OA are together greater than PC;

that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the O^{ce};

∴ PA is the greatest of all such lines.

(ii) In the $\triangle$ OPD, the sides OP, PD are together greater than OD.

But OD = OB, being radii;

$\therefore$ OP, PD are together greater than OB.

Take away the common part OP;

then PD is greater than PB.

Similarly any other st. line drawn from P to the $\odot^{ce}$ may be shewn to be greater than PB;

$\therefore$ PB is the least of all such lines.

(iii) In the $\triangle^s$ POC, POD,

because $\begin{cases} \text{PO is common,} \\ \text{OC = OD, being radii,} \\ \text{but the } \angle \text{ POC is greater than the } \angle \text{ POD;} \end{cases}$

$\therefore$ PC is greater than PD. *Theor.* 19.

Q.E.D.

EXERCISES.

(*Miscellaneous.*)

1. *All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.*

2 If two circles which intersect are cut by a straight line parallel to the common chord, shew that the parts of it intercepted between the circumferences are equal.

3. If two circles cut one another, any two parallel straight lines drawn through the points of intersection to cut the circles are equal.

4. If two circles cut one another, any two straight lines drawn through a point of section, making equal angles with the common chord, and terminated by the circumferences, are equal.

5. Two circles of diameters 74 and 40 inches respectively have a common chord 2 feet in length: find the distance between their centres.

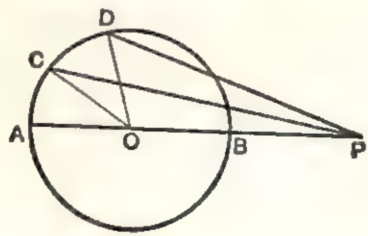
Draw the figure (1 cm. to represent 10") and verify your result by measurement.

6. Draw two circles of radii 1·0" and 1·7", and with their centres 2·1" apart. Find by calculation, and by measurement, the length of the common chord, and its distance from the two centres.

*Theorem 37. [Euclid III. 8.]

If from any external point straight lines are drawn to the circumference of a circle, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre.

And of any other two such lines, the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from any external point P let the lines PBA, PC, PD be drawn to the O^{ce}, so that PBA passes through the centre O, and so that the ∠POC subtended by PC at the centre is greater than the ∠POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*
(ii) PB *is the least,*
(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the △POC, the sides PO, OC are together greater than PC.

But OC = OA, being radii;
∴ PO, OA are together greater than PC;
that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the O^{ce};

that is, PA is the greatest of all such lines

(ii) In the $\triangle$POD, the sides PD, DO are together greater than PO.

But OD = OB, being radii;

$\therefore$ the remainder PD is greater than the remainder PB.

Similarly any other st. line drawn from P to the $\odot^{ce}$ may be shewn to be greater than PB;

that is, PB is the least of all such lines.

(iii) In the $\triangle^s$ POC, POD,

because $\begin{cases} \text{PO is common,} \\ \text{OC = OD, being radii;} \\ \text{but the } \angle \text{POC is greater than the } \angle \text{POD;} \end{cases}$

$\therefore$ PC is greater than PD. *Theor.* 19.

Q.E.D.

EXERCISES.

(*Miscellaneous.*)

1. Find the greatest and least straight lines which have one extremity on each of two given circles which do not intersect.

2. If from any point on the circumference of a circle straight lines are drawn to the circumference, the greatest is that which passes through the centre; and of any two such lines the greater is that which subtends the greater angle at the centre.

3. Of all straight lines drawn through a point of intersection of two circles, and terminated by the circumferences, the greatest is that which is parallel to the line of centres.

4. Draw on squared paper any two circles which have their centres on the x-axis, and cut at the point (8, −11). Find the coordinates of their other point of intersection.

5. Draw on squared paper two circles with centres at the points (15, 0) and (−6, 0) respectively, and cutting at the point (0, 8). Find the lengths of their radii, and the coordinates of their other point of intersection.

6. Draw an isosceles triangle OAB with an angle of 80° at its vertex O. With centre O and radius OA draw a circle, and on its circumference take any number of points P, Q, R, ... on the same side of AB as the centre. Measure the angles subtended by the chord AB at the points P, Q, R, ... Repeat the same exercise with any other given angle at O. What inference do you draw?

ON ANGLES IN SEGMENTS, AND ANGLES AT THE CENTRES AND CIRCUMFERENCES OF CIRCLES.

THEOREM 38. [Euclid III. 20.]

The angle at the centre of a circle is double of an angle at the circumference standing on the same arc.

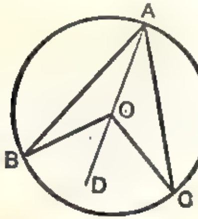


Fig. 1.

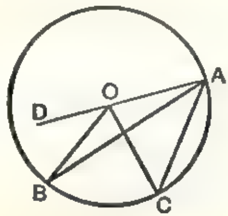


Fig. 2.

Let ABC be a circle, of which O is the centre; and let BOC be the angle at the centre, and BAC an angle at the $\odot^{ce}$, standing on the same arc BC.

It is required to prove that the $\angle$ BOC *is twice the* $\angle$ BAC.

Join AO, and produce it to D.

Proof. In the $\triangle$ OAB, because OB = OA,

$\therefore$ the $\angle$ OAB = the $\angle$ OBA.

$\therefore$ the sum of the $\angle^s$ OAB, OBA = twice the $\angle$ OAB.

But the ext. $\angle$ BOD = the sum of the $\angle^s$ OAB, OBA;

$\therefore$ the $\angle$ BOD = twice the $\angle$ OAB.

Similarly the $\angle$ DOC = twice the $\angle$ OAC.

$\therefore$, adding these results in Fig. 1, and taking the difference in Fig. 2, it follows in each case that

the $\angle$ BOC = twice the $\angle$ BAC. Q.E.D.

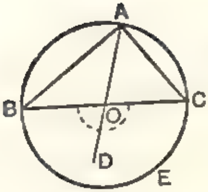


Fig. 3.

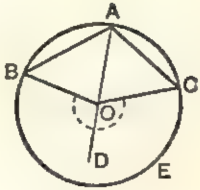


Fig. 4.

Obs. If the arc BEC, on which the angles stand, is a semi-circumference, as in Fig. 3, the ∠BOC at the centre is a *straight angle*; and if the arc BEC is greater than a semi-circumference, as in Fig. 4, the ∠BOC at the centre is *reflex.* But the proof for Fig. 1 applies without change to both these cases, shewing that whether the given arc is greater than, equal to, or less than a semi-circumference,

the ∠BOC = *twice the* ∠BAC, *on the same arc* BEC.

DEFINITIONS.

A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.

NOTE. The chord of a segment is sometimes called its base.

An **angle in a segment** is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

We have seen in Theorem 32 that a circle may be drawn through *any three* points not in a straight line. But it is only under certain conditions that a circle can be drawn through more than three points.

DEFINITION. If four or more points are so placed that a circle may be drawn through them, they are said to be **concyclic.**

THEOREM 39. [Euclid III. 21.]

Angles in the same segment of a circle are equal.

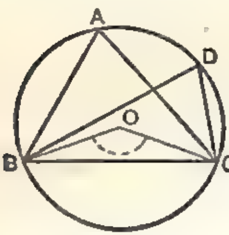


Fig. 1.

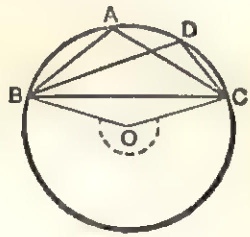


Fig. 2.

Let BAC, BDC be angles in the same segment BADC of a circle, whose centre is O.

It is required to prove that the $\angle$ BAC = *the* $\angle$ BDC.

Join BO, OC.

Proof. Because the $\angle$ BOC is at the centre, and the $\angle$ BAC at the O^{ce}, standing on the same arc BC,

$\therefore$ the $\angle$ BOC = twice the $\angle$ BAC. *Theor.* 38.

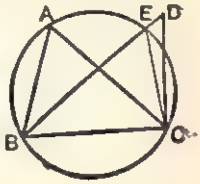
Similarly the $\angle$ BOC = twice the $\angle$ BDC.

$\therefore$ the $\angle$ BAC = the $\angle$ BDC. Q.E.D.

NOTE. The given segment may be greater than a semicircle as in Fig. 1, or less than a semicircle as in Fig. 2: in the latter case the angle BOC will be reflex. But by virtue of the extension of Theorem 38, given on the preceding page, the above proof applies equally to both figures.

CONVERSE OF THEOREM 39.

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.



Let BAC, BDC be two equal angles standing on the same base BC, and on the same side of it.

It is required to prove that A *and* D *lie on an arc of a circle having* BC *as its chord.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts BD or BD produced at the point E.

Join EC.

Proof. Then the $\angle$BAC = the $\angle$BEC in the same segment.

But, by hypothesis, the $\angle$BAC = the $\angle$BDC;

$\therefore$ the $\angle$BEC = the $\angle$BDC;

which is impossible unless E coincides with D;

$\therefore$ the circle through B, A, C must pass through D.

COROLLARY. *The locus of the vertices of triangles drawn on the same side of a given base, and with equal vertical angles, is an arc of a circle.*

EXERCISES ON THEOREM 39.

1. In Fig. 1, if the angle BDC is 74°, find the number of degrees in each of the angles BAC, BOC, OBC.

2. In Fig. 2, let BD and CA intersect at X. If the angle DXC = 40°, and the angle XCD = 25°, find the number of degrees in the angle BAC and in the reflex angle BOC.

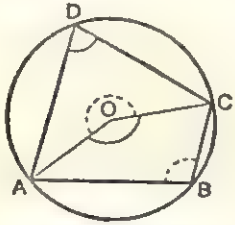
3. In Fig. 1, if the angles CBD, BCD are respectively 43° and 82°, find the number of degrees in the angles BAC, OBD, OCD.

4. Shew that in Fig. 2 the angle OBC is always less than the angle BAC by a right angle.

[For further Exercises on Theorem 39 see page 170.]

THEOREM 40. [Euclid III. 22.]

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilateral inscribed in the ⊙ABC.

It is required to prove that

(i) *the* ∠ˢ ADC, ABC *together = two rt. angles.*

(ii) *the* ∠ˢ BAD, BCD *together = two rt. angles.*

Suppose O is the centre of the circle.

Join OA, OC.

Proof. Since the ∠ADC at the ⊙ᶜᵉ = half the ∠AOC at the centre, standing on the same arc ABC;
and the ∠ABC at the ⊙ᶜᵉ = half the reflex ∠AOC at the centre, standing on the same arc ADC;

∴ the ∠ˢ ADC, ABC together = half the sum of the ∠AOC and the reflex ∠AOC.

But these angles make up four rt. angles.

∴ the ∠ˢ ADC, ABC together = two rt. angles.

Similarly the ∠ˢ BAD, BCD together = two rt. angles.

Q.E.D.

NOTE. The results of Theorems 39 and 40 should be carefully compared.

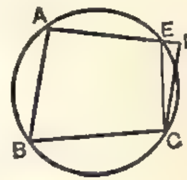
From Theorem 39 we learn that angles in the *same* segment are *equal.*

From Theorem 40 we learn that angles in *conjugate* segments are *supplementary.*

DEFINITION. A quadrilateral is called **cyclic** when a circle can be drawn through its four vertices.

Converse of Theorem 40.

✓ *If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.*



Let ABCD be a quadrilateral in which the opposite angles at B and D are supplementary.

It is required to prove that the points A, B, C, D *are concyclic.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts AD or AD produced in the point E.

Join EC.

Proof. Then since ABCE is a cyclic quadrilateral,
$\therefore$ the $\angle$AEC is the supplement of the $\angle$ABC.
But, by hypothesis, the $\angle$ADC is the supplement of the $\angle$ABC;
$\therefore$ the $\angle$AEC = the $\angle$ADC;
which is impossible unless E coincides with D.
$\therefore$ the circle which passes through A, B, C must pass through D:
that is, A, B, C, D are concyclic. Q.E.D.

EXERCISES ON THEOREM 40.

1. In a circle of 1·6″ radius inscribe a quadrilateral ABCD, making the angle ABC equal to 126°. Measure the remaining angles, and hence verify in this case that opposite angles are supplementary.

2. Prove Theorem 40 by the aid of Theorems 39 and 16, after first joining the opposite vertices of the quadrilateral.

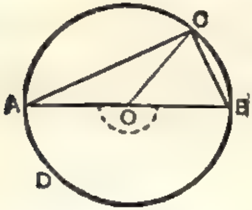
3. If a circle can be described about a parallelogram, the parallelogram must be rectangular.

4. ABC is an isosceles triangle, and XY is drawn parallel to the base BC cutting the sides in X and Y: shew that the four points B, C, X, Y lie on a circle.

5. *If one side of a cyclic quadrilateral is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.*

THEOREM 41. [Euclid III. 31.]

The angle in a semi-circle is a right angle.



Let ADB be a circle of which AB is a diameter and O the centre; and let C be any point on the semi-circumference ACB.

It is required to prove that the ∠ ACB *is a rt. angle.*

1st Proof. The ∠ ACB at the $\odot^{ce}$ is half the *straight angle* AOB at the centre, standing on the same arc ADB;

and a *straight angle* = two rt. angles:

∴ the ∠ ACB is a rt. angle. Q.E.D.

2nd Proof. Join OC.

Then because OA = OC,

∴ the ∠ OCA = the ∠ OAC. *Theor.* 5.

And because OB = OC,

∴ the ∠ OCB = the ∠ OBC.

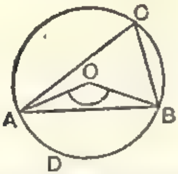
∴ the whole ∠ ACB = the ∠ OAC + the ∠ OBC.

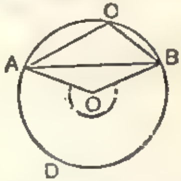
But the three angles of the △ ACB together = two rt. angles;

∴ the ∠ ACB = one-half of two rt. angles

= one rt. angle. Q.E.D.

COROLLARY. *The angle in a segment greater than a semi-circle is acute; and the angle in a segment less than a semi-circle is obtuse.*





The ∠ACB at the O^ce is half the ∠AOB at the centre, on the same arc ADB.

(i) If the segment ACB is greater than a semi-circle, then ADB is a *minor* arc;

∴ the ∠AOB is *less* than two rt. angles;
∴ the ∠ACB is *less* than one rt. angle.

(ii) If the segment ACB is less than a semi-circle, then ADB is a *major* arc;

∴ the ∠AOB is *greater* than two rt. angles;
∴ the ∠ACB is *greater* than one rt. angle.

EXERCISES ON THEOREM 41.

1. *A circle described on the hypotenuse of a right-angled triangle as diameter, passes through the opposite angular point.*

2. Two circles intersect at A and B; and through A two diameters AP, AQ are drawn, one in each circle: shew that the points P, B, Q are collinear.

3. A circle is described on one of the equal sides of an isosceles triangle as diameter. Shew that it passes through the middle point of the base.

4. Circles described on any two sides of a triangle as diameters intersect on the third side, or the third side produced.

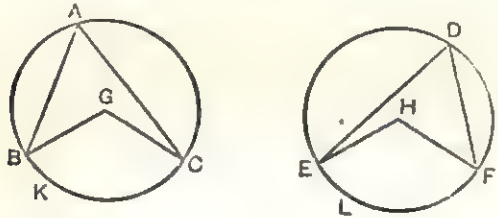
5. A straight rod of given length slides between two straight rulers placed at right angles to one another; find the locus of its middle point.

6. *Find the locus of the middle points of chords of a circle drawn through a fixed point. Distinguish between the cases when the given point is within, on, or without the circumference.*

DEFINITION. A sector of a circle is a figure bounded by two radii and the arc intercepted between them.

THEOREM 42. [Euclid III. 26.]

In equal circles, arcs which sublend equal angles, either at the centres or at the circumferences, are equal.



Let ABC, DEF be equal circles, and let the ∠ BGC = the ∠ EHF at the centres; and consequently

the ∠ BAC = the ∠ EDF at the ○^ces. *Theor.* 38.

It is required to prove that the arc BKC = *the arc* ELF.

Proof. Apply the ⊙ ABC to the ⊙ DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the ∠ BGC = the ∠ EHF,

∴ GC will fall along HF.

And because the circles have equal radii, B will fall on E, and C on F, and the circumferences of the circles will coincide entirely.

∴ the arc BKC must coincide with the arc ELF;

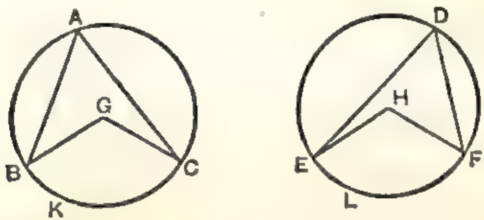
∴ the arc BKC = the arc ELF. Q.E.D.

COROLLARY. *In equal circles sectors which have equal angles are equal.*

Obs. It is clear that any theorem relating to arcs, angles, and chords in *equal* circles must also be true in the *same* circle.

THEOREM 43. [Euclid III 27.]

In equal circles angles, either at the centres or at the circumferences, which stand on equal arcs are equal.



Let ABC, DEF be equal circles;
and let the arc BKC = the arc ELF.

It is required to prove that
the ∠ BGC = *the* ∠ EHF *at the centres;*
also the ∠ BAC = *the* ∠ EDF *at the* ⊙^{ces}

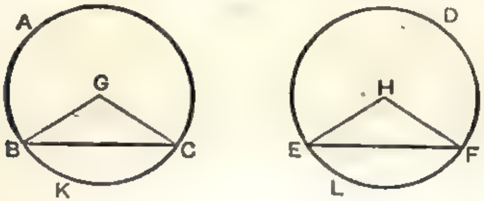
Proof. Apply the ⊙ABC to the ⊙DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the circles have equal radii,
∴ B falls on E, and the two ⊙^{ces} coincide entirely.
And, by hypothesis, the arc BKC = the arc ELF.
∴ C falls on F, and consequently GC on HF;
∴ the ∠ BGC = the ∠ EHF.

And since the ∠ BAC at the ⊙^{ce} = half the ∠ BGC at the centre;
and likewise the ∠ EDF = half the ∠ EHF;
∴ the ∠ BAC = the ∠ EDF. Q.E.D.

Theorem 44. [Euclid III. 28.]

In equal circles, arcs which are cut off by equal chords are equal, the major arc equal to the major arc, and the minor to the minor.



Let ABC, DEF be equal circles whose centres are G and H; and let the chord BC = the chord EF.

It is required to prove that

the major arc BAC = *the major arc* EDF,

and *the minor arc* BKC = *the minor arc* ELF.

Join BG, GC, EH, HF.

Proof. In the $\triangle^s$ BGC, EHF,

because { BG = EH, being radii of equal circles
GC = HF, for the same reason,
and BC = EF, by hypothesis;

∴ the ∠ BGC = the ∠ EHF; *Theor.* 7.

∴ the arc BKC = the arc ELF; *Theor.* 42.

and these are the minor arcs.

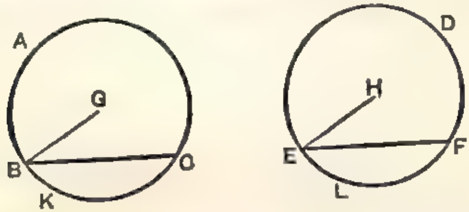
But the whole $\odot^{ce}$ ABKC = the whole $\odot^{ce}$ DELF;

∴ the remaining arc BAC = the remaining arc EDF:

and these are the major arcs. Q.E.D.

THEOREM 45. [Euclid III. 29.]

In equal circles chords which cut off equal arcs are equal.



Let ABC, DEF be equal circles whose centres are G and H; and let the arc BKC = the arc ELF.

It is required to prove that the chord BC = *the chord* EF.

Join BG, EH.

Proof. Apply the ⊙ABC to the ⊙DEF, so that G falls on H and GB along HE.

Then because the circles have equal radii,
∴ B falls on E, and the $\bigcirc^{ces}$ coincide entirely.

And because the arc BKC = the arc ELF,
∴ C falls on F.

∴ the chord BC coincides with the chord EF;
∴ the chord BC = the chord EF. Q.E.D.

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EXERCISES ON ANGLES IN A CIRCLE.

1. P is any point on the arc of a segment of which AB is the chord. Shew that the sum of the angles PAB, PBA is constant.

2. PQ and RS are two chords of a circle intersecting at X: prove that the triangles PXS, RXQ are equiangular to one another.

3. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that PQ subtends a constant angle at B.

4. Two circles intersect at A and B; and through A any two straight lines PAQ, XAY are drawn terminated by the circumferences; shew that the arcs PX, QY subtend equal angles at B.

5. P is any point on the arc of a segment whose chord is AB: and the angles PAB, PBA are bisected by straight lines which intersect at O. Find the locus of the point O.

6. *If two chords intersect within a circle, they form an angle equal to that at the centre, subtended by half the sum of the arcs they cut off.*

7. *If two chords intersect without a circle, they form an angle equal to that at the centre subtended by half the difference of the arcs they cut off.*

8. The sum of the arcs cut off by two chords of a circle at right angles to one another is equal to the semi-circumference.

9. *If* AB *is a fixed chord of a circle and* P *any point on one of the arcs cut off by it, then the bisector of the angle* APB *cuts the conjugate arc in the same point for all positions of* P.

10. AB, AC are any two chords of a circle; and P, Q are the middle points of the minor arcs cut off by them; if PQ is joined, cutting AB in X and AC in Y, shew that AX = AY.

11. A triangle ABC is inscribed in a circle, and the bisectors of the angles meet the circumference at X, Y, Z. Show that the angles of the triangle XYZ are respectively

$$90^\circ-\frac{A}{2},\quad 90^\circ-\frac{B}{2},\quad 90^\circ-\frac{C}{2}.$$

12. Two circles intersect at A and B; and through these points lines are drawn from any point P on the circumference of one of the circles: shew that when produced they intercept on the other circumference an arc which is constant for all positions of P.

13. The straight lines which join the extremities of parallel chords in a circle (i) towards the same parts, (ii) towards opposite parts, are equal.

14. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn: shew that the chord PX is equal to the chord QY.

15. Through the points of intersection of two circles two parallel straight lines are drawn terminated by the circumferences: shew that the straight lines which join their extremities towards the same parts are equal.

16. Two equal circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that BP = BQ.

17. ABC is an isosceles triangle inscribed in a circle, and the bisectors of the base angles meet the circumference at X and Y. Shew that the figure BXAYC must have four of its sides equal.

What relation must subsist among the angles of the triangle ABC, in order that the figure BXAYC may be equilateral?

18. ABCD is a cyclic quadrilateral, and the opposite sides AB, DC are produced to meet at P, and CB, DA to meet at Q: if the circles circumscribed about the triangles PBC, QAB intersect at R, shew that the points P, R, Q are collinear.

19. P, Q, R *are the middle points of the sides of a triangle, and* X *is the foot of the perpendicular let fall from one vertex on the opposite side: shew that the four points* P, Q, R, X *are concyclic.*

[See page 64, Ex. 2: also Prob. 10, p. 83.]

20. *Use the preceding exercise to shew that the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides, are concyclic.*

21. If a series of triangles are drawn standing on a fixed base, and having a given vertical angle, shew that the bisectors of the vertical angles all pass through a fixed point.

22. ABC is a triangle inscribed in a circle, and E the middle point of the arc subtended by BC on the side remote from A: if through E a diameter ED is drawn, shew that the angle DEA is half the difference of the angles at B and C.

TANGENCY.

Definitions and First Principles.

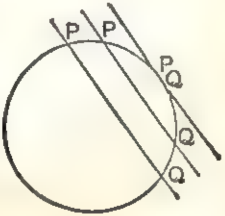
1. A **secant** of a circle is a straight line of indefinite length which cuts the circumference at two points.

2. If a secant moves in such a way that the two points in which it cuts the circle continually approach one another, then in the ultimate position when these two points become one, the secant becomes a **tangent** to the circle, and is said to **touch** it at the point at which the two intersections coincide. This point is called the **point of contact.**

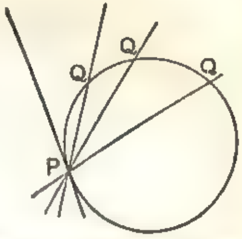
For instance:

(i) Let a secant cut the circle at the points P and Q, and suppose it to recede from the centre, moving always parallel to its original position; then the two points P and Q will clearly approach one another and finally coincide.

In the ultimate position when P and Q become one point, the straight line becomes a tangent to the circle at that point.



(ii) Let a secant cut the circle at the points P and Q, and suppose it to be turned about the point P so that while P remains fixed, Q moves on the circumference nearer and nearer to P. Then the line PQ in its ultimate position, when Q coincides with P, is a tangent at the point P.



Since a secant can cut a circle at *two* points only, it is clear that a tangent can have only one point in common with the circumference, namely the point of contact, at which two points of section coincide. Hence we may define a tangent as follows:

3. A **tangent** to a circle is a straight line which meets the circumference at one point only; and though produced indefinitely does not cut the circumference.

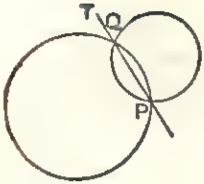


Fig. 1.

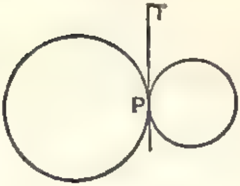


Fig. 2.

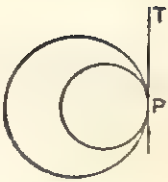


Fig. 3.

4. Let two circles intersect (as in Fig. 1) in the points P and Q, and let one of the circles turn about the point P, which remains fixed, in such a way that Q continually approaches P. Then in the ultimate position, when Q coincides with P (as in Figs. 2 and 3), the circles are said to touch one another at P.

Since two circles cannot intersect in more than *two* points, two circles which touch one another cannot have more than *one* point in common, namely the point of contact at which the two points of section coincide. Hence circles are said to touch one another when they meet, but do not cut one another.

NOTE. When each of the circles which meet is *outside the other*, as in Fig. 2, they are said to touch one another externally, or to have external contact: when one of the circles is *within the other*, as in Fig. 3, the first is said to touch the other **internally**, or to have **internal contact with it.**

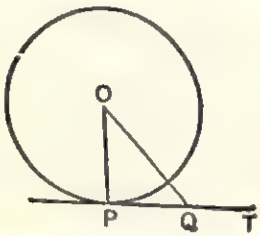
INFERENCE FROM DEFINITIONS 2 AND 4.

If in Fig. 1, TQP is a common chord of two circles one of which is made to turn about P, then when Q is brought into coincidence with P, the line TP passes through two coincident points on each circle, as in Figs. 2 and 3, and therefore becomes a tangent to each circle. Hence

Two circles which touch one another have a common tangent at their point of contact.

THEOREM 46.

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.



Let PT be a tangent at the point P to a circle whose centre is O.

It is required to prove that PT *is perpendicular to the radius* OP.

Proof. Take *any* point Q in PT, and join OQ.

Then since PT is a tangent, every point in it except P is outside the circle.

$\therefore$ OQ is greater than the radius OP.

And this is true for every point Q in PT;

$\therefore$ OP is the shortest distance from O to PT.

Hence OP is perp. to PT. *Theor.* 12, *Cor.* 1.

Q.E.D.

COROLLARY 1. Since there can be only one perpendicular to OP at the point P, it follows that *one and only one tangent can be drawn to a circle at a given point on the circumference.*

COROLLARY 2. Since there can be only one perpendicular to PT at the point P, it follows that *the perpendicular to a tangent at its point of contact passes through the centre.*

COROLLARY 3. Since there can be only one perpendicular from O to the line PT, it follows that *the radius drawn perpendicular to the tangent passes through the point of contact.*

THEOREM 46. [By the Method of Limits.]

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.

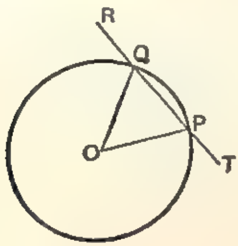


Fig. 1.

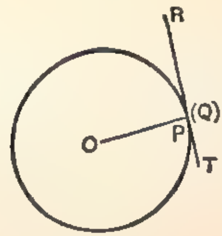


Fig. 2.

Let P be a point on a circle whose centre is O.

It is required to prove that the tangent at P *is perpendicular to the radius* OP.

Let RQPT (Fig. 1) be a secant cutting the circle at Q and P.

Join OQ, OP.

Proof. Because OP = OQ,

∴ the ∠OQP = the ∠OPQ;

∴ the supplements of these angles are equal;

that is, the ∠OQR = the ∠OPT,

and this is true however near Q is to P.

Now let the secant QP be turned about the point P so that Q continually approaches and finally coincides with P; then in the ultimate position,

(i) the secant RT becomes the tangent at P, } Fig. 2,
(ii) OQ coincides with OP; }

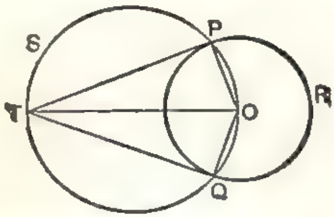
and therefore the equal ∠ˢ OQR, OPT become adjacent,

∴ OP is perp. to RT. Q.E.D.

NOTE. The method of proof employed here is known as the Method of Limits.

THEOREM 47.

Two tangents can be drawn to a circle from an external point.



Let PQR be a circle whose centre is O, and let T be an external point.

It is required to prove that there can be two tangents drawn to the circle from T.

Join OT, and let TSO be the circle on OT as diameter.

This circle will cut the ⊙PQR in two points, since T is without, and O is within, the ⊙PQR. Let P and Q be these points.

Join TP, TQ; OP, OQ.

Proof. Now each of the ∟ˢ TPO, TQO, being in a semicircle, is a rt. angle;

∴ TP, TQ are perp. to the radii OP, OQ respectively.

∴ TP, TQ are tangents at P and Q. *Theor.* 46.

Q.E.D.

COROLLARY. *The two tangents to a circle from an external point are equal, and subtend equal angles at the centre.*

For in the △ˢ TPO, TQO,

because $\begin{cases} \text{the } \angle^s \text{ TPO, TQO are } \textit{right angles}, \\ \text{the hypotenuse TO is common,} \\ \text{and OP} = \text{OQ, being radii;} \end{cases}$

∴ TP = TQ,

and the ∠TOP = the ∠TOQ. *Theor.* 18.

EXERCISES ON THE TANGENT.

(Numerical and Graphical.)

1. Draw two concentric circles with radii 5·0 cm. and 3·0 cm. Draw a series of chords of the former to touch the latter. Calculate and measure their lengths, and account for their being equal.

2. In a circle of radius 1·0" draw a number of chords each 1·6" in length. Shew that they all touch a concentric circle, and find its radius.

3. The diameters of two concentric circles are respectively 10·0 cm. and 5·0 cm.: find to the nearest millimetre the length of any chord of the outer circle which touches the inner, and check your work by measurement.

4. In the figure of Theorem 47, if OP=5", TO=13", find the length of the tangents from T. Draw the figure (scale 2 cm. to 5"), and measure to the nearest degree the angles subtended at O by the tangents.

5. The tangents from T to a circle whose radius is 0·7" are each 2·4" in length. Find the distance of T from the centre of the circle. Draw the figure and check your result graphically.

(Theoretical.)

6. *The centre of any circle which touches two intersecting straight lines must lie on the bisector of the angle between them.*

7. AB and AC are two tangents to a circle whose centre is O; shew that AO bisects the chord of contact BC at right angles.

8. If PQ is joined in the figure of Theorem 47, shew that the angle PTQ is double the angle OPQ.

9. Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

10. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

11. *Find the locus of the centres of all circles which touch a given straight line at a given point.*

12. Find the locus of the centres of all circles which touch each of two parallel straight lines.

13. *Find the locus of the centres of all circles which touch each of two intersecting straight lines of unlimited length.*

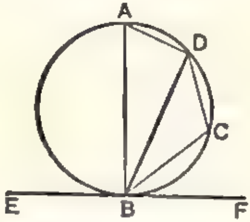
14. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

State and prove the converse theorem.

15. If a quadrilateral is described about a circle, the angles subtended at the centre by any two opposite sides are supplementary.

THEOREM 49. [Euclid III. 32.]

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.



Let EF touch the ⊙ABC at B, and let BD be a chord drawn from B, the point of contact.

It is required to prove that

(i) *the* ∠FBD = *the angle in the alternate segment* BAD;

(ii) *the* ∠EBD = *the angle in the alternate segment* BCD.

Let BA be the diameter through B, and C any point in the arc of the segment which does not contain A.

Join AD, DC, CB.

Proof. Because the ∠ADB in a semicircle is a rt. angle,

∴ the ∠s DBA, BAD together = a rt. angle.

But since EBF is a tangent, and BA a diameter,

∴ the ∠FBA is a rt. angle.

∴ the ∠FBA = the ∠s DBA, BAD together.

Take away the common ∠DBA,

then the ∠FBD = the ∠BAD, which is in the alternate segment.

Again because ABCD is a cyclic quadrilateral,

∴ the ∠BCD = the supplement of the ∠BAD

= the supplement of the ∠FBD

= the ∠EBD;

∴ the ∠EBD = the ∠BCD, which is in the alternate segment.

Q.E.D.

EXERCISES ON THEOREM 49.

1. In the figure of Theorem 49, if the $\angle$FBD=72°, write down the values of the $\angle$s BAD, BCD, EBD.

2. Use this theorem to shew that tangents to a circle from an external point are equal.

3. Through A, the point of contact of two circles, chords APQ, AXY are drawn: shew that PX and QY are parallel.

Prove this (i) for internal, (ii) for external contact.

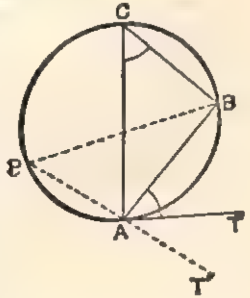
4. AB is the common chord of two circles, one of which passes through O, the centre of the other: prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

5. Two circles intersect at A and B; and through P, any point on one of them, straight lines PAC, PBD are drawn to cut the other at C and D: shew that CD is parallel to the tangent at P.

6. If from the point of contact of a tangent to a circle a chord is drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

EXERCISES ON THE METHOD OF LIMITS.

1. *Prove Theorem 49 by the Method of Limits.*



[Let ACB be a segment of a circle of which AB is the chord; and let PAT′ be any *secant* through A. Join PB.

Then the $\angle$BCA=the $\angle$BPA; *Theor.* 39.

and this is true *however near* P *approaches to* A.

If P moves up to coincidence with A, then the secant PAT′ becomes the tangent AT, and the $\angle$BPA becomes the $\angle$BAT.

∴, ultimately, the $\angle$BAT=the $\angle$BCA, in the alt. segment.]

2. From Theorem 31, prove by the Method of Limits that

The straight line drawn perpendicular to the diameter of a circle at its extremity is a tangent.

3. Deduce Theorem 48 from the property that *the line of centres bisects a common chord at right angles.*

4. Deduce Theorem 49 from Ex. 5, page 163.

5. Deduce Theorem 46 from Theorem 41.

PROBLEMS.

GEOMETRICAL ANALYSIS.

Hitherto the Propositions of this text-book have been arranged **Synthetically**, that is to say, by *building up known results* in order to obtain a *new* result.

But this arrangement, though convincing as an argument, in most cases affords little clue as to the way in which the construction or proof *was discovered.* We therefore draw the student's attention to the following hints.

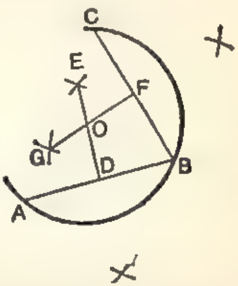
In attempting to solve a problem begin by *assuming* the required result; then by working backwards, trace the consequences of the assumption, and try to ascertain its dependence on some condition or known theorem which suggests the necessary construction. If this attempt is successful, the steps of the argument may in general be re-arranged in reverse order, and the construction and proof presented in a synthetic form.

This unravelling of the conditions of a proposition in order to trace it back to some earlier principle on which it depends, is called **geometrical analysis**: it is the natural way of attacking the harder types of exercises, and it is especially useful in solving problems.

Although the above directions do not amount to a *method,* they often furnish a very effective mode of *searching for a suggestion.* The approach by analysis will be illustrated in some of the following problems. [See Problems 23, 28, 29.]

PROBLEM 20.

Given a circle, or an arc of a circle, to find its centre.



Let ABC be an arc of a circle whose centre is to be found.

Construction. Take two chords AB, BC, and bisect them at right angles by the lines DE, FG, meeting at O. *Prob.* 2.

Then O is the required centre.

Proof. Every point in DE is equidistant from A and B. *Prob.* 14.

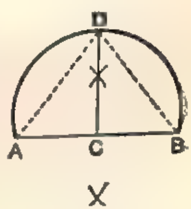
And every point in FG is equidistant from B and C.

∴ O is equidistant from A, B, and C.

∴ O is the centre of the circle ABC. *Theor.* 33.

PROBLEM 21.

To bisect a given arc.



Let ADB be the given arc to be bisected.

Construction. Join AB, and bisect it at right angles by CD meeting the arc at D. *Prob.* 2.

Then the arc is bisected at D.

Proof. Join DA, DB.

Then every point on CD is equidistant from A and B; *Prob.* 14.

∴ DA = DB;

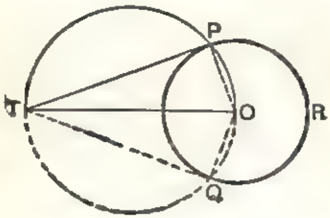
∴ the ∠ DBA = the ∠ DAB; *Theorem* 6.

∴ the arcs, which subtend these angles at the ⊙^{ce}, are equal;

that is, the arc DA = the arc DB.

PROBLEM 22.

To draw a tangent to a circle from a given external point.



Let PQR be the given circle, with its centre at C; and let T be the point from which a tangent is to be drawn.

Construction. Join TO, and on it describe a semi-circle TPO to cut the circle at P.

Join TP.

Then TP is the required tangent.

Proof. Join OP.

Then since the ∠ TPO, being in a semi-circle, is a rt. angle,

∴ TP is at right angles to the radius OP.

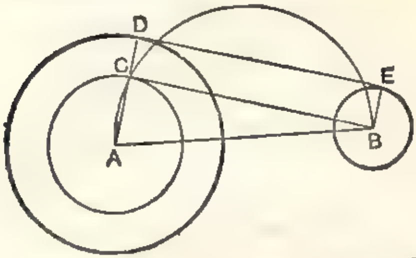
∴ TP is a tangent at P. *Theor.* 46.

Since the semi-circle may be described on either side of TO, a second tangent TQ can be drawn from T, as shewn in the figure.

NOTE. Suppose the point T to approach the given circle, then the angle PTQ gradually increases. When T reaches the circumference, the angle PTQ becomes a *straight angle*, and the two tangents coincide. When T enters the circle, no tangent can be drawn. [See *Obs.* p. 94.]

PROBLEM 23.

To draw a common tangent to two circles.



Let A be the centre of the greater circle, and a its radius; and let B be the centre of the smaller circle, and b its radius.

Analysis. Suppose DE to touch the circles at D and E. Then the radii AD, BE are both perp. to DE, and therefore par^l to one another.

Now if BC were drawn par^l to DE, then the fig. DB would be a rectangle, so that $CD = BE = b$.

And if AD, BE are *on the same side of* AB,

then $AC = a - b$, and the $\angle ACB$ is a rt. angle.

These hints enable us to draw BC *first*, and thus lead to the following construction.

Construction. With centre A, and radius equal to the *difference* of the radii of the given circles, describe a circle, and draw BC to touch it.

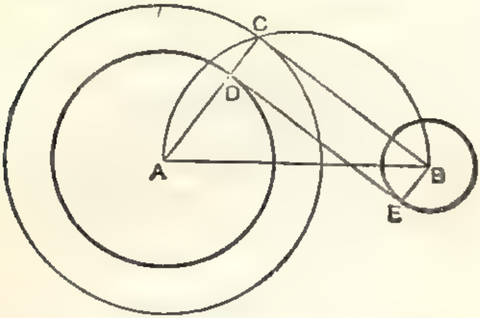
Join AC, and produce it to meet the circle (A) at D.

Through B draw the radius BE par^l to AD and in the *same sense*. Join DE.

Then DE is a common tangent to the given circles.

Obs. Since two tangents, such as BC, can in general be drawn from B to the circle of construction, this method will furnish two common tangents to the given circles. These are called the **direct common tangents**.

PROBLEM 23. (*Continued.*)



Again, if the circles are *external to one another* two more common tangents may be drawn.

Analysis. In this case we may suppose DE to touch the circles at D and E so that the radii AD, BE fall *on opposite sides of* AB.

Then BC, drawn par' to the supposed common tangent DE, would meet AD *produced* at C; and we should now have

AC = AD + DC = $a + b$; and, as before, the ∠ ACB is a rt. angle.

Hence the following construction.

Construction. With centre A, and radius equal to the *sum* of the radii of the given circles, describe a circle, and draw BC to touch it.

Then proceed as in the first case, but draw BE in the sense *opposite* to AD.

Obs. As before, two tangents may be drawn from B to the circle of construction; hence two common tangents may be thus drawn to the given circles. These are called the **transverse common tangents.**

[We leave as an exercise to the student the arrangement of the proof in synthetic form.]

EXERCISES ON COMMON TANGENTS.

(*Numerical and Graphical.*)

1. How many common tangents can be drawn in each of the following cases?

(i) when the given circles intersect;
(ii) when they have external contact;
(iii) when they have internal contact.

Illustrate your answer by drawing two circles of radii 1·4″ and 1·0″ respectively,

(i) with 1·0″ between the centres;
(ii) with 2·4″ between the centres;
(iii) with 0·4″ between the centres;
(iv) with 3·0″ between the centres.

Draw the common tangents in each case, and note where the general construction fails, or is modified.

2. Draw two circles with radii 2·0″ and 0·8″, placing their centres 2·0″ apart. Draw the common tangents, and find their lengths between the points of contact, both by calculation and by measurement.

3. Draw all the common tangents to two circles whose centres are 1·8″ apart and whose radii are 0·6″ and 1·2″ respectively. Calculate and measure the length of the direct common tangents.

4. Two circles of radii 1·7″ and 1·0″ have their centres 2·1″ apart. Draw their common tangents and find their lengths. Also find the length of the common chord. Produce the common chord and shew by measurement that it bisects the common tangents.

5. Draw two circles with radii 1·6″ and 0·8″ and with their centres 3·0″ apart. Draw all their common tangents.

6. Draw the direct common tangents to two equal circles.

(*Theoretical.*)

7. If the two direct, or the two transverse, common tangents are drawn to two circles, the parts of the tangents intercepted between the points of contact are equal.

8. If four common tangents are drawn to two circles external to one another, shew that the two direct, and also the two transverse, tangents intersect on the line of centres.

9. Two given circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that PQ subtends a right angle at the point A.

On the Construction of Circles.

In order to draw a circle we must know (i) the position of the centre, (ii) the length of the radius.

(i) To find the position of the centre, two conditions are needed, each giving a locus on which the centre must lie; so that the one or more points in which the two loci intersect are possible positions of the required centre, as explained on page 93.

(ii) The position of the centre being thus fixed, the radius is determined if we know (or can find) any point on the circumference.

Hence in order to draw a circle *three* independent data are required.

For example, we may draw a circle if we are given

(i) *three* points on the circumference;

or (ii) *three* tangent lines;

or (iii) one point on the circumference, one tangent, and its point of contact.

It will however often happen that more than one circle can be drawn satisfying three given conditions.

Before attempting the constructions of the next Exercise the student should make himself familiar with the following loci.

(i) *The locus of the centres of circles which pass through two given points.*

(ii) *The locus of the centres of circles which touch a given straight line at a given point.*

(iii) *The locus of the centres of circles which touch a given circle at a given point.*

(iv) *The locus of the centres of circles which touch a given straight line, and have a given radius.*

(v) *The locus of the centres of circles which touch a given circle, and have a given radius.*

(vi) *The locus of the centres of circles which touch two given straight lines.*

EXERCISES.

1. Draw a circle to pass through three given points.

2. If a circle touches a given line PQ at a point A, on what line must its centre lie?

If a circle passes through two given points A and B, on what line must its centre lie?

Hence draw a circle to touch a straight line PQ at the point A, and to pass through another given point B.

3. If a circle touches a given circle whose centre is C at the point A, on what line must its centre lie?

Draw a circle to touch the given circle (C) at the point A, and to pass through a given point B.

4. A point P is 4·5 cm. distant from a straight line AB. Draw two circles of radius 3·2 cm. to pass through P and to touch AB.

5. Given two circles of radius 3·0 cm. and 2·0 cm. respectively, their centres being 6·0 cm. apart; draw a circle of radius 3·5 cm. to touch each of the given circles externally.

How many solutions will there be? What is the radius of the smallest circle that touches each of the given circles externally?

6. If a circle touches two straight lines OA, OB, on what line must its centre lie?

Draw OA, OB making an angle of 76°, and describe a circle of radius 1·2" to touch both lines.

7. Given a circle of radius 3·5 cm., with its centre 5·0 cm. from a given straight line AB; draw two circles of radius 2·5 cm. to touch the given circle and the line AB.

8. Devise a construction for drawing a circle to touch each of two parallel straight lines and a transversal.

Shew that two such circles can be drawn, and that they are equal.

9. Describe a circle to touch a given circle, and also to touch a given straight line at a given point. [See page 311.]

10. Describe a circle to touch a given straight line, and to touch a given circle at a given point.

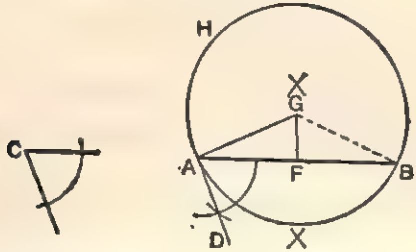
11. Shew how to draw a circle to touch each of three given straight lines of which no two are parallel.

How many such circles can be drawn?

[Further Examples on the Construction of Circles will be found on pp. 246, 311.]

Problem 24.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given st. line, and C the given angle.

It is required to describe on AB *a segment of a circle containing an angle equal to* C.

Construction. At A in BA, make the ∠BAD equal to the ∠C.

From A draw AG perp. to AD.

Bisect AB at rt. angles by FG, meeting AG in G. *Prob.* 2.

Proof. Join GB.

Now every point in FG is equidistant from A and B; *Prob.* 14.

∴ GA = GB.

With centre G, and radius GA, draw a circle, which must pass through B, and touch AD at A. *Theor.* 46.

Then the segment AHB, alternate to the ∠BAD, contains an angle equal to C. *Theor.* 49.

Note. In the particular case when the given angle is a rt. angle, the segment required will be the semi-circle on AB as diameter. [Theorem 41.]

COROLLARY. *To cut off from a given circle a segment containing a given angle, it is enough to draw a tangent to the circle, and from the point of contact to draw a chord making with the tangent an angle equal to the given angle.*

It was proved on page 161 that

The locus of the vertices of triangles which stand on the same base and have a given vertical angle, is the arc of the segment standing on this base, and containing an angle equal to the given angle.

The following Problems are derived from this result by the Method of Intersection of Loci [page 93].

EXERCISES.

1. *Describe a triangle on a given base having a given vertical angle and having its vertex on a given straight line.*

2. *Construct a triangle having given the base, the vertical angle, and*
   (i) *one other side.*
   (ii) *the altitude.*
   (iii) *the length of the median which bisects the base.*
   (iv) *the foot of the perpendicular from the vertex to the base.*

3. *Construct a triangle having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.*

[Let AB be the base, X the given point in it, and K the given angle. On AB describe a segment of a circle containing an angle equal to K; complete the O^{ce} by drawing the arc APB. Bisect the arc APB at P: join PX, and produce it to meet the O^{ce} at C. Then ABC is the required triangle.]

4. *Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.*

[Let AB be the given base, K the given angle, and H a line equal to the sum of the sides. On AB describe a segment containing an angle equal to K, also another segment containing an angle equal to half the $\angle$K. With centre A, and radius H, describe a circle cutting the arc of the latter segment at X and Y. Join AX (or AY) cutting the arc of the first segment at C. Then ABC is the required triangle.]

5. *Construct a triangle having given the base, the vertical angle, and the difference of the remaining sides.*

CIRCLES IN RELATION TO RECTILINEAL FIGURES.

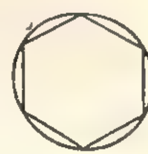
DEFINITIONS.

1. A **Polygon** is a rectilineal figure bounded by more than four sides.

A Polygon of	*five* sides	is called a	Pentagon,
"	*six* sides	"	Hexagon,
"	*seven* sides	"	Heptagon,
"	*eight* sides	"	Octagon,
"	*ten* sides	"	Decagon,
"	*twelve* sides	"	Dodecagon,
"	*fifteen* sides	"	Quindecagon.

2. A Polygon is **Regular** when all its sides are equal, and all its angles are equal.

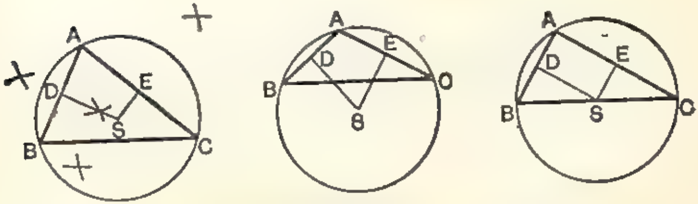
3. A rectilineal figure is said to be **inscribed** in a circle, when all its angular points are on the circumference of the circle; and a circle is said to be **circumscribed** about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure.



4. A circle is said to be **inscribed** in a rectilineal figure, when the circumference of the circle is touched by each side of the figure; and a rectilineal figure is said to be **circumscribed** about a circle, when each side of the figure is a tangent to the circle.

PROBLEM 25.

To circumscribe a circle about a given triangle.



Let ABC be the triangle, about which a circle is to be drawn.

Construction. Bisect AB and AC at rt. angles by DS and ES, meeting at S. *Prob.* 2.

Then S is the centre of the required circle.

Proof. Now every point in DS is equidistant from A and B; *Prob.* 14.

and every point in ES is equidistant from A and C;

∴ S is equidistant from A, B, and C.

With centre S, and radius SA describe a circle; this will pass through B and C, and is, therefore, the required circ n-circle.

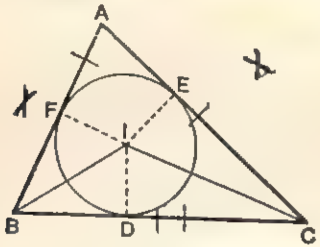
Obs. It will be found that if the given triangle is acute-angled, the centre of the circum-circle falls within it: if t is a right-angled triangle, the centre falls on the hypotenuse: if it is an obtuse-angled triangle, the centre falls without the triangle.

NOTE. From page 94 it is seen that if S is joined to the middle point of BC, then the joining line is perpendicular to BC.

Hence the perpendiculars drawn to the sides of a triangle from their middle points are concurrent, the point of intersection being the centre of the circle circumscribed about the triangle.

PROBLEM 26.

To inscribe a circle in a given triangle.



Let ABC be the triangle, in which a circle is to be inscribed.

Construction. Bisect the $\angle$s ABC, ACB by the st. lines BI, CI, which intersect at I. *Prob.* 1.

Then I is the centre of the required circle.

Proof. From I draw ID, IE, IF perp. to BC, CA, AB.

Then every point in BI is equidistant from BC, BA; *Prob.* 15.

$\therefore$ ID = IF.

And every point in CI is equidistant from CB, CA;

$\therefore$ ID = IE.

$\therefore$ ID, IE, IF are all equal.

With centre I and radius ID draw a circle;
this will pass through the points E and F.

Also the circle will touch the sides BC, CA, AB,
because the angles at D, E, F are right angles.

$\therefore$ the $\odot$ DEF is inscribed in the $\triangle$ ABC.

NOTE. From II., p. 96 it is seen that if AI is joined, then AI bisects the angle BAC: hence it follows that

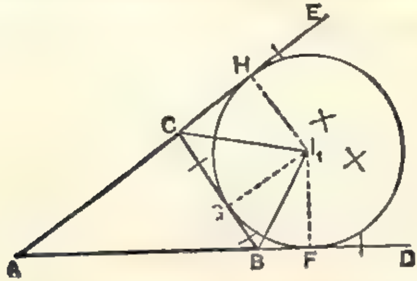
The bisectors of the angles of a triangle are concurrent, the point of intersection being the centre of the inscribed circle.

DEFINITION.

A circle which touches one side of a triangle and the other two sides produced is called an **escribed** circle of the triangle.

PROBLEM 27.

To draw an escribed circle of a given triangle.



Let ABC be the given triangle of which the sides AB, AC are produced to D and E.

It is required to describe a circle touching BC, *and* AB, AC *produced.*

Construction. Bisect the ∠ˢ CBD, BCE by the st. lines BI_1, CI_1 which intersect at I_1.

Then I_1 is the centre of the required circle.

Proof. From I_1 draw I_1F, I_1G, I_1H perp. to AD, BC, AE.

Then every point in BI_1 is equidistant from BD, BC; *Prob.* 15.

$\therefore$ $I_1F = I_1G$.

Similarly $I_1G = I_1H$.

$\therefore$ I_1F, I_1G, I_1H are all equal.

With centre I_1 and radius I_1F describe a circle;
this will pass through the points G and H.
Also the circle will touch AD, BC, and AE,
because the angles at F, G, H are rt. angles.
$\therefore$ the ⊙FGH is an escribed circle of the △ABC.

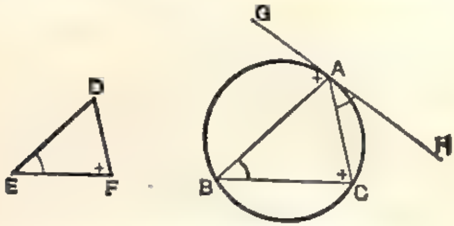
NOTE 1. It is clear that every triangle has three escribed circles. Their centres are known as the **Ex-centres**.

NOTE 2. It may be shewn, as in II., page 96, that if AI_1 is joined, then AI_1 bisects the angle BAC: hence it follows that

The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent, the point of intersection being the centre of an escribed circle.

PROBLEM 28.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

Analysis. A $\triangle$ ABC, equiangular to the $\triangle$ DEF, is inscribed in the circle, if from any point A on the $\bigcirc^{ce}$ two chords AB, AC can be so placed that, on joining BC, the $\angle$ B = the $\angle$ E, and the $\angle$ C = the $\angle$ F; for then the $\angle$ A = the $\angle$ D. *Theor.* 16.

Now the $\angle$ B, in the segment ABC, suggests the *equal* angle between the chord AC and the tangent at its extremity (*Theor.* 49.); so that, if at A we draw the tangent GAH,

then the $\angle$ HAC = the $\angle$ E;

and similarly, the $\angle$ GAB = the $\angle$ F.

Reversing these steps, we have the following construction.

Construction. At any point A on the $\bigcirc^{ce}$ of the $\odot$ ABC draw the tangent GAH. *Prob.* 22.

At A make the $\angle$ GAB equal to the $\angle$ F,

and make the $\angle$ HAC equal to the $\angle$ E.

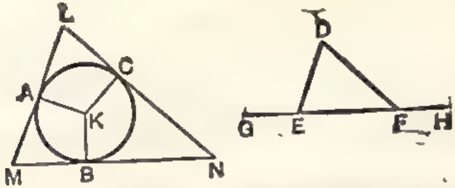
Join BC.

Then ABC is the required triangle.

NOTE. In drawing the figure on a larger scale the student should shew the construction lines for the tangent GAH and for the angles GAB, HAC. A similar remark applies to the next Problem.

PROBLEM 29.

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

Analysis. Suppose LMN to be a circumscribed triangle in which the $\angle$ M = the $\angle$ E, the $\angle$ N = the $\angle$ F, and consequently, the $\angle$ L = the $\angle$ D.

Let us consider the radii KA, KB, KC, drawn to the points of contact of the sides; for the tangents LM, MN, NL could be drawn if we knew the relative positions of KA, KB, KC, that is, if we knew the $\angle$s BKA, BKC.

Now from the quad BKAM, since the $\angle$s B and A are rt. $\angle$s,

$$\text{the } \angle \text{BKA} = 180^\circ - \text{M} = 180^\circ - \text{E};$$

similarly $\quad$ the $\angle$ BKC $= 180^\circ -$ N $= 180^\circ -$ F.

Hence we have the following construction.

Construction. Produce EF both ways to G and H.

Find K the centre of the $\odot$ ABC,
and draw any radius KB.
At K make the $\angle$ BKA equal to the $\angle$ DEG;
and make the $\angle$ BKC equal to the $\angle$ DFH.

Through A, B, C draw LM, MN, NL perp. to KA, KB, KC.
Then LMN is the required triangle.

[The student should now arrange the proof synthetically.]

EXERCISES.

On Circles and Triangles.

(Inscriptions and Circumscriptions.)

1. In a circle of radius 5 cm. inscribe an equilateral triangle; and about the same circle circumscribe a second equilateral triangle. In each case state and justify your construction.

2. Draw an equilateral triangle on a side of 8 cm., and find by calculation and measurement (to the nearest millimetre) the radii of the inscribed, circumscribed, and escribed circles.

Explain why the second and third radii are respectively double and treble of the first.

3. Draw triangles from the following data:

$$\text{(i) } a=2{\cdot}5'', \quad B=66^\circ, \quad C=50^\circ;$$
$$\text{(ii) } a=2{\cdot}5'', \quad B=72^\circ, \quad C=44^\circ;$$
$$\text{(iii) } a=2{\cdot}5'', \quad B=41^\circ, \quad C=23^\circ.$$

Circumscribe a circle about each triangle, and measure the radii to the nearest hundredth of an inch. Account for the three results being the same, by comparing the vertical angles.

4. In a circle of radius 4 cm. inscribe an equilateral triangle. Calculate the length of its side to the nearest millimetre; and verify by measurement.

Find the area of the inscribed equilateral triangle, and shew that it is one quarter of the circumscribed equilateral triangle.

5. In the triangle ABC, if I is the centre, and r the length of the radius of the in-circle, shew that

$$\triangle IBC=\tfrac{1}{2}ar; \quad \triangle ICA=\tfrac{1}{2}br; \quad \triangle IAB=\tfrac{1}{2}cr.$$

Hence prove that $\triangle ABC=\frac{1}{2}(a+b+c)r$.

Verify this formula by measurements for a triangle whose sides are 9 cm., 8 cm., and 7 cm.

6. If r_1 is the radius of the ex-circle opposite to A, prove that

$$\triangle ABC=\tfrac{1}{2}(b+c-a)r_1.$$

If $a=5$ cm., $b=4$ cm., $c=3$ cm., verify this result by measurement.

7. Find by measurement the circum-radius of the triangle ABC in which $a=6{\cdot}3$ cm., $b=3{\cdot}0$ cm., and $c=5{\cdot}1$ cm.

Draw and measure the perpendiculars from A, B, C to the opposite sides. If their lengths are represented by p_1, p_2, p_3, verify the following statement:

$$\text{circum-radius}=\frac{bc}{2p_1}=\frac{ca}{2p_2}=\frac{ab}{2p_3}.$$

EXERCISES.

On Circles and Squares.

(Inscriptions and Circumscriptions.)

1. Draw a circle of radius 1·5", and find a construction for inscribing a square in it.

Calculate the length of the side to the nearest hundredth of an inch, and verify by measurement.

Find the area of the inscribed square.

2. Circumscribe a square about a circle of radius 1·5", shewing all lines of construction.

Prove that the area of the square circumscribed about a circle is double that of the inscribed square.

3. Draw a square on a side of 7·5 cm., and state a construction for inscribing a circle in it.

Justify your construction by considerations of symmetry.

4. Circumscribe a circle about a square whose side is 6 cm.

Measure the diameter to the nearest millimetre, and test your drawing by calculation.

5. In a circle of radius 1·8" inscribe a rectangle of which one side measures 3·0". Find the approximate length of the other side.

Of all rectangles inscribed in the circle shew that the square has the greatest area.

6. A square and an equilateral triangle are inscribed in a circle. If a and b denote the lengths of their sides, shew that

$$3a^2=2b^2.$$

7. ABCD is a square inscribed in a circle, and P is any point on the arc AD: shew that the side AD subtends at P an angle three times as great as that subtended at P by any one of the other sides.

(Problems. State your construction, and give a theoretical proof.)

8. Circumscribe a rhombus about a given circle.

9. Inscribe a square in a given square ABCD, so that one of its angular points shall be at a given point X in AB.

10. In a given square inscribe the square of minimum area.

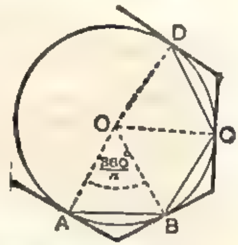
11. Describe (i) a circle, (ii) a square about a given rectangle.

12. Inscribe (i) a circle, (ii) a square in a given quadrant.

ON CIRCLES AND REGULAR POLYGONS.

PROBLEM 30.

To draw a regular polygon (i) in (ii) about *a given circle.*



Let AB, BC, CD, ... be consecutive sides of a regular polygon inscribed in a circle whose centre is O.

Then AOB, BOC, COD, ... are congruent isosceles triangles. And if the polygon has n sides, each of the $\angle$s AOB, BOC, COD, ... $= \frac{360°}{n}$.

(i) Thus to inscribe a polygon of n sides in a given circle, draw an angle AOB at the centre equal to $\frac{360°}{n}$. This gives the length of a side AB; and chords equal to AB may now be set off round the circumference. The resulting figure will clearly be equilateral and equiangular.

(ii) To circumscribe a polygon of n sides about the circle, the points A, B, C, D, ... must be determined as before, and tangents drawn to the circle at these points. The resulting figure may readily be proved equilateral and equiangular.

NOTE. This method gives a *strict geometrical construction* only when the angle $\frac{360°}{n}$ can be drawn with ruler and compasses.

EXERCISES.

1. Give strict constructions for inscribing in a circle (radius 4 cm.) (i) a regular hexagon; (ii) a regular octagon; (iii) a regular dodecagon.

2. About a circle of radius 1·5" circumscribe

(i) a regular hexagon; (ii) a regular octagon.

Test the constructions by measurement, and justify them by proof.

3. An equilateral triangle and a regular hexagon are inscribed in a given circle, and a and b denote the lengths of their sides: prove that

(i) area of triangle $= \frac{1}{2}$(area of hexagon); (ii) $a^2 = 3b^2$.

4. By means of your protractor inscribe a regular heptagon in a circle of radius 2". Calculate and measure one of its angles; and measure the length of a side.

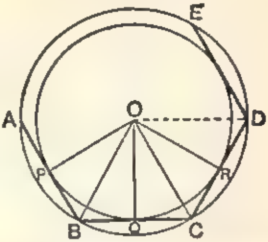
PROBLEM 31.

To draw a circle (i) in (ii) about *a regular polygon.*

Let AB, BC, CD, DE, ... be consecutive sides of a regular polygon of n sides.

Bisect the $\angle$ˢ ABC, BCD by BO, CO meeting at O.

Then O is the centre both of the inscribed and circumscribed circle.



Outline of Proof. Join OD; and from the congruent $\triangle$ˢ OCB, OCD, shew that OD bisects the $\angle$CDE. Hence we conclude that

All the bisectors of the angles of the polygon meet at O.

(i) Prove that OB = OC = OD = ... ; from Theorem 6.
Hence O is the circum-centre.

(ii) Draw OP, OQ, OR, ... perp. to AB, BC, CD, ...
Prove that OP = OQ = OR = ... ; from the congruent $\triangle$ˢ OBP, OBQ, ...
Hence O is the in-centre.

EXERCISES.

1. Draw a regular hexagon on a side of 2·0″. Draw the inscribed and circumscribed circles. Calculate and measure their diameters to the nearest hundredth of an inch.

2. Shew that the area of a regular hexagon inscribed in a circle is three-fourths of that of the circumscribed hexagon.

Find the area of a hexagon inscribed in a circle of radius 10 cm. to the nearest tenth of a sq. cm.

3. If ABC is an isosceles triangle inscribed in a circle, having each of the angles B and C double of the angle A; shew that BC is a side of a regular pentagon inscribed in the circle.

4. On a side of 4 cm. construct (without protractor)

(i) a regular hexagon; (ii) a regular octagon.

In each case find the approximate area of the figure.

THE CIRCUMFERENCE OF A CIRCLE.

By experiment and measurement it is found that the length of the circumference of a circle is roughly $3\frac{1}{7}$ times the length of its diameter: that is to say

$$\frac{circumference}{diameter} = 3\tfrac{1}{7} \text{ nearly};$$

and it can be proved that this is the same for all circles.

A more correct value of this ratio is found by theory to be 3·1416; while correct to 7 places of decimals it is 3·1415926. Thus the value $3\frac{1}{7}$ (or 3·14*28*) is too great, and correct to 2 places only.

The ratio which the circumference of any circle bears to its diameter is denoted by the Greek letter π; so that

$$circumference = diameter \times \pi.$$

Or, if r denotes the radius of the circle,

$$circumference = 2r \times \pi = 2\pi r;$$

where to π we are to give one of the values $3\frac{1}{7}$, 3·1416, or 3·1415926, according to the degree of accuracy required in the final result.

NOTE. The theoretical methods by which π is evaluated to any required degree of accuracy cannot be explained at this stage, but its value may be easily verified by experiment to two decimal places.

For example: round a cylinder wrap a strip of paper so that the ends overlap. At any point in the overlapping area prick a pin through both folds. Unwrap and straighten the strip, then measure the distance between the pin holes: this gives the length of the circumference. Measure the diameter, and divide the first result by the second.

Ex. 1. From these data find and record the value of π.

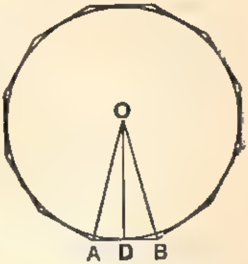
Find the mean of the three results.

CIRCUMFERENCE.	DIAMETER.	VALUE OF π.
16·0 cm.	5·1 cm.	
8·8"	2·8"	
13·5"	4·3"	

Ex. 2. A fine thread is wound evenly round a cylinder, and it is found that the length required for 20 complete turns is 75·4". The diameter of the cylinder is 1·2": find roughly the value of π.

Ex. 3. A bicycle wheel, 28" in diameter, makes 400 revolutions in travelling over 977 yards. From this result estimate the value of π.

THE AREA OF A CIRCLE.



Let AB be a side of a polygon of n sides circumscribed about a circle whose centre is O and radius r. Then we have

Area of polygon

$= n . \triangle AOB$

$= n . \frac{1}{2} AB \times OD$

$= \frac{1}{2} . nAB \times r$

$= \frac{1}{2}$ (*perimeter of polygon*) $\times r$;

and this is true however many sides the polygon may have.

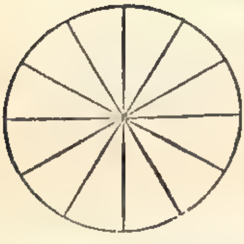
Now if the number of sides is increased without limit, the perimeter and area of the polygon may be made to differ from the circumference and area of the circle by quantities smaller than any that can be named; hence ultimately

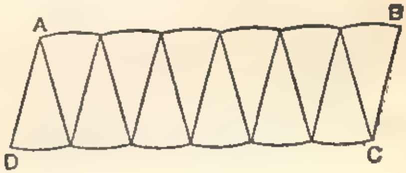
Area of circle $= \frac{1}{2}$. *circumference* $\times r$

$= \frac{1}{2} . 2\pi r \times r$

$= \pi r^2.$

ALTERNATIVE METHOD.





Suppose the circle divided into any even number of sectors having equal central angles: denote the number of sectors by n.

Let the sectors be placed side by side as represented in the diagram;

then the area of the circle = the area of the fig. ABCD;

and this is true however great n may be.

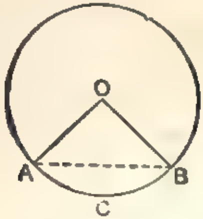
Now as the number of sectors is increased, each arc is decreased; so that (i) the outlines AB, CD tend to become *straight*, and

(ii) the angles at D and B tend to become rt. angles.

Thus when n is increased without limit, the fig. ABCD ultimately becomes a *rectangle*, whose length is the *semi-circumference of the circle*, and whose breadth is its *radius*.

$$\therefore \text{ Area of circle} = \tfrac{1}{2}\,.\,\text{circumference} \times \text{radius}$$
$$= \tfrac{1}{2}\,.\,2\pi r \times r = \pi r^2.$$

THE AREA OF A SECTOR.



If two radii of a circle make an angle of 1°, they cut off

(i) an arc whose length $= \frac{1}{360}$ of the circumference;

and (ii) a sector whose area $= \frac{1}{360}$ of the circle;

$\therefore$ if the angle AOB contains D degrees, then

(i) *the arc* AB $= \dfrac{D}{360}$ *of the circumference*;

(ii) *the sector* AOB $= \dfrac{D}{360}$ *of the area of the circle*

$= \dfrac{D}{360}$ *of* ($\tfrac{1}{2}$ *circumference* × *radius*)

$= \tfrac{1}{2}$. *arc* AB × *radius*.

THE AREA OF A SEGMENT.

The area of a minor segment is found by subtracting from the corresponding sector the area of the triangle formed by the chord and the radii. Thus

Area of segment ABC = *sector* OACB − *triangle* AOB.

The area of a major segment is most simply found by subtracting the area of the corresponding minor segment from the area of the circle.

EXERCISES.

[*In each case choose the value of π so as to give a result of the assigned degree of accuracy.*]

1. Find to the nearest millimetre the circumferences of the circles whose radii are (i) 4·5 cm. (ii) 100 cm.

2. Find to the nearest hundredth of a square inch the areas of the circles whose radii are (i) 2·3″. (ii) 10·6″.

3. Find to two places of decimals the circumference and area of a circle inscribed in a square whose side is 3·6 cm.

4. In a circle of radius 7·0 cm. a square is described: find to the nearest square centimetre the difference between the areas of the circle and the square.

5. Find to the nearest hundredth of a square inch the area of the circular ring formed by two concentric circles whose radii are 5·7″ and 4·3″.

6. Shew that the area of a ring lying between the circumferences of two concentric circles is equal to the area of a circle whose radius is the length of a tangent to the inner circle from any point on the outer.

7. A rectangle whose sides are 8·0 cm. and 6·0 cm. is inscribed in a circle. Calculate to the nearest tenth of a square centimetre the total area of the four segments outside the rectangle.

8. Find to the nearest tenth of an inch the side of a square whose area is equal to that of a circle of radius 5″.

9. A circular ring is formed by the circumference of two concentric circles. The area of the ring is 22 square inches, and its width is 1·0″; taking π as $\frac{22}{7}$, find approximately the radii of the two circles.

10. Find to the nearest hundredth of a square inch the difference between the areas of the circumscribed and inscribed circles of an equilateral triangle each of whose sides is 4″.

11. Draw on squared paper two circles whose centres are at the points (1·5″, 0) and (0, ·8″), and whose radii are respectively ·7″ and 1·0″. Prove that the circles touch one another, and find approximately their circumferences and areas.

12. Draw a circle of radius 1·0″ having the point (1·6″, 1·2″) as centre. Also draw two circles with the origin as centre and of radii 1·0″ and 3·0″ respectively. Show that each of the last two circles touches the first.

EXERCISES.

On the Inscribed, Circumscribed, and Escribed Circles of a Triangle.

(Theoretical.)

1. Describe a circle to touch two parallel straight lines and a third straight line which meets them. Shew that two such circles can be drawn, and that they are equal.

2. *Triangles which have equal bases and equal vertical angles have equal circumscribed angles.*

3. ABC is a triangle, and I, S are the centres of the inscribed and circumscribed circles; if A, I, S are collinear, shew that AB = AC.

4. The sum of the diameters of the inscribed and circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

5. If the circle inscribed in the triangle ABC touches the sides at D, E, F; show that the angles of the triangle DEF are respectively

$$90-\frac{A}{2},\ 90-\frac{B}{2},\ 90-\frac{C}{2}.$$

6. If I is the centre of the circle inscribed in the triangle ABC, and I_1 the centre of the escribed circle which touches BC; shew that I, B, I_1, C are concyclic.

7. In any triangle the difference of two sides is equal to the difference of the segments into which the third side is divided at the point of contact of the inscribed circle.

8. In the triangle ABC, I and S are the centres of the inscribed and circumscribed circles: shew that IS subtends at A an angle equal to half the difference of the angles at the base of the triangle.

Hence shew that if AD is drawn perpendicular to BC, then AI is the bisector of the angle DAS.

9. The diagonals of a quadrilateral ABCD intersect at O: shew that the centres of the circles circumscribed about the four triangles AOB, BOC, COD, DOA are at the angular points of a parallelogram.

10. In any triangle ABC, if I is the centre of the inscribed circle, and if AI is produced to meet the circumscribed circle at O; shew that O is the centre of the circle circumscribed about the triangle BIC.

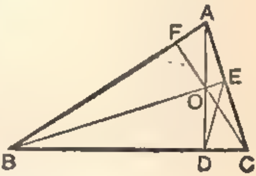
11. Given the base, altitude, and the radius of the circumscribed circle; construct the triangle.

12. Three circles whose centres are A, B, C touch one another externally two by two at D, E, F: shew that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.

THEOREMS AND EXAMPLES ON CIRCLES AND TRIANGLES.

THE ORTHOCENTRE OF A TRIANGLE.

I. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



In the △ABC, let AD, BE be the perp^s drawn from A and B to the opposite sides; and let them intersect at O.

Join CO; and produce it to meet AB at F.

It is required to shew that CF *is perp. to* AB.

Join DE.

Then, because the ∠^{s} OEC, ODC are rt. angles,

∴ the points O, E, C, D are concyclic:

∴ the ∠DEC = the ∠DOC, in the same segment
= the vert. opp. ∠FOA.

Again, because the ∠^{s} AEB, ADB are rt. angles,

∴ the points A, E, D, B are concyclic:

∴ the ∠DEB = the ∠DAB, in the same segment.

∴ the sum of the ∠^{s} FOA, FAO = the sum of the ∠^{s} DEC, DEB
= a rt. angle:

∴ the remaining ∠AFO = a rt. angle: *Theor.* 16.

that is, CF is perp. to AB.

Hence the three perp^s AD, BE, CF meet at the point O.

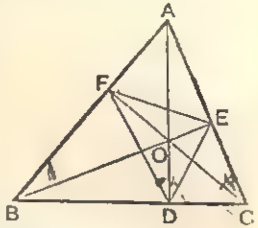
Q.E.D.

DEFINITIONS.

(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its orthocentre.

(ii) The triangle formed by joining the feet of the perpendiculars is called the pedal or orthocentric triangle.

II. *In an acute-angled triangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle through which they pass.*



In the acute-angled $\triangle$ ABC, let AD, BE, CF be the perp^s drawn from the vertices to the opposite sides, meeting at the orthocentre O; and let DEF be the pedal triangle.

It is required to prove that

AD, BE, CF *bisect respectively the* $\angle$^s FDE, DEF, EFD.

It may be shewn, as in the last theorem, that the points O, D, C, E are concyclic;

$\therefore$ the $\angle$ODE = the $\angle$OCE, in the same segment.

Similarly the points O, D, B, F are concyclic;

$\therefore$ the $\angle$ODF = the $\angle$OBF, in the same segment.

But the $\angle$OCE = the $\angle$OBF, each being the comp^t of the $\angle$BAC.

$\therefore$ the $\angle$ODE = the $\angle$ODF.

Similarly it may be shewn that the $\angle$^s DEF, EFD are bisected by BE and CF.

Q.E.D.

Corollary. (i) *Every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet.*

For the $\angle$EDC = the comp^t of the $\angle$ODE
= the comp^t of the $\angle$OCE
= the $\angle$BAC.

Similarly it may be shewn that the $\angle$FDB = the $\angle$BAC,

$\therefore$ the $\angle$EDC = the $\angle$FDB = the $\angle$A.

In like manner it may be proved that

the $\angle$DEC = the $\angle$FEA = the $\angle$B,
and the $\angle$DFB = the $\angle$EFA = the $\angle$C.

Corollary. (ii) *The triangles* DEC, AEF, DBF *are equiangular to one another and to the triangle* ABC.

Note. If the angle BAC is *obtuse*, then the perpendiculars BE, CF bisect *externally* the corresponding angles of the pedal triangle.

EXERCISES.

1. *If* O *is the orthocentre of the triangle* ABC *and if the perpendicular* AD *is produced to meet the circum-circle in* G, *prove that* OD = DG.

2. *In an acute-angled triangle the three sides are the external bisectors of the angles of the pedal triangle: and in an obtuse-angled triangle the sides containing the obtuse angle are the internal bisectors of the corresponding angles of the pedal triangle.*

3. *If* O *is the orthocentre of the triangle* ABC, *shew that the angles* BOC, BAC *are supplementary.*

4. *If* O *is the orthocentre of the triangle* ABC, *then any one of the four points* O, A, B, C *is the orthocentre of the triangle whose vertices are the other three.*

5 *The three circles which pass through two vertices of a triangle and its orthocentre are each equal to the circum-circle of the triangle.*

6 D, E are taken on the circumference of a semi-circle described on a given straight line AB: the chords AD, BE and AE, BD intersect (produced if necessary) at F and G: shew that FG is perpendicular to AB.

7 ABC is a triangle, O is its orthocentre, and AK a diameter of the circum circle: shew that BOCK is a parallelogram.

8. The orthocentre of a triangle is joined to the middle point of the base, and the joining line is produced to meet the circum-circle: prove that it will meet it at the same point as the diameter which passes through the vertex.

9. The perpendicular from the vertex of a triangle on the base, and the straight line joining the orthocentre to the middle point of the base, are produced to meet the circum-circle at P and Q: shew that PQ is parallel to the base.

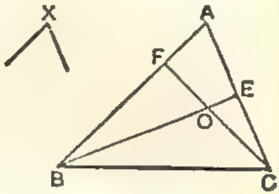
10. *The distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the centre of the circum-circle to the opposite side.*

11. Three circles are described each passing through the orthocentre of a triangle and two of its vertices: shew that the triangle formed by joining their centres is equal in all respects to the original triangle.

12. Construct a triangle, having given a vertex, the orthocentre, and the centre of the circum-circle.

LOCI.

III. *Given the base and vertical angle of a triangle, find the locus of its orthocentre.*



Let BC be the given base, and X the given angle; and let BAC be any triangle on the base BC, having its vertical $\angle A$ equal to the $\angle X$.

Draw the perp^s BE, CF, intersecting at the orthocentre O.

It is required to find the locus of O.

Proof. Since the $\angle^s$ OFA, OEA are rt. angles,

$\therefore$ the points O, F, A, E are concyclic;

$\therefore$ the $\angle FOE$ is the supplement of the $\angle A$:

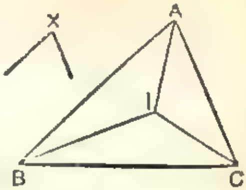
$\therefore$ the vert. opp. $\angle BOC$ is the supplement of the $\angle A$.

But the $\angle A$ is constant, being always equal to the $\angle X$;

$\therefore$ its supplement is constant;

that is, the $\triangle BOC$ has a fixed base, and constant vertical angle; hence the locus of its vertex O is the arc of a segment of which BC is the chord.

IV. *Given the base and vertical angle of a triangle, find the locus of the in-centre.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given $\angle X$; and let AI, BI, CI be the bisectors of its angles. Then I is the in-centre.

It is required to find the locus of I.

Proof. Denote the angles of the $\triangle ABC$ by A, B, C; and let the $\angle BIC$ be denoted by I.

Then from the $\triangle BIC$,

(i) $\quad I + \frac{1}{2}B + \frac{1}{2}C =$ two rt. angles; *Theor.* 16.

and from the $\triangle ABC$,

$A + B + C =$ two rt. angles;

(ii) so that $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C =$ one rt. angle.

$\therefore$, taking the differences of the equals in (i) and (ii),

$I - \frac{1}{2}A =$ one rt. angle:

or, $I =$ one rt. angle $+ \frac{1}{2}A$.

But A is constant, being always equal to the $\angle X$;

$\therefore$ I is constant:

$\therefore$ the locus of I is the arc of a segment on the fixed chord BC.

EXERCISES ON LOCI.

1. Given the base BC and the vertical angle A of a triangle; find the locus of the ex-centre opposite A.

2. Through the extremities of a given straight line AB any two parallel straight lines AP, BQ are drawn; find the locus of the intersection of the bisectors of the angles PAB, QBA.

3. Find the locus of the middle points of chords of a circle drawn through a fixed point.

Distinguish between the cases when the given point is within, on, or without the circumference.

4. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

5. Find the locus of the intersection of straight lines which pass through two fixed points on a circle and intercept on its circumference an arc of constant length.

6. A and B are two fixed points on the circumference of a circle, and PQ is any diameter: find the locus of the intersection of PA and QB.

7. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle: find the locus of P.

8. AB is a fixed chord of a circle, and AC is a moveable chord passing through A: if the parallelogram CB is completed, find the locus of the intersection of its diagonals.

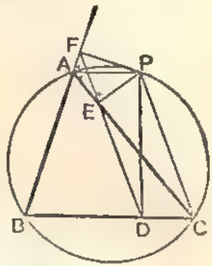
9 A straight rod PQ slides between two rulers placed at right angles to one another, and from its extremities PX, QX are drawn perpendicular to the rulers: find the locus of X.

10. Two circles intersect at A and B, and through P, any point on the circumference of one of them, two straight lines PA, PB are drawn, and produced if necessary, to cut the other circle at X and Y: find the locus of the intersection of AY and BX.

11. Two circles intersect at A and B; HAK is a fixed straight line drawn through A and terminated by the circumferences, and PAQ is any other straight line similarly drawn: find the locus of the intersection of HP and QK.

SIMSON'S LINE.

V. *The feet of the perpendiculars drawn to the three sides of a triangle from any point on its circum-circle are collinear.*



Let P be any point on the circum-circle of the △ ABC; and let PD, PE, PF be the perps. drawn from P to the sides.

It is required to prove that the points D, E, F *are collinear.*

Join FE and ED:

then FE and ED will be shewn to be in the same straight line.

Join PA, PC.

Proof. Because the ∠ˢ PEA, PFA are rt. angles,

∴ the points P, E, A, F are concyclic:

∴ the ∠PEF = the ∠PAF, in the same segment
= the suppᵗ of the ∠PAB
= the ∠PCD,

since the points A, P, C, B are concyclic.

Again because the ∠ˢ PEC, PDC are rt. angles,

∴ the points P, E, D, C are concyclic.

∴ the ∠PED = the suppᵗ of the ∠PCD
= the suppᵗ of the ∠PEF.

∴ FE and ED are in one st. line.

Obs. The line FED is known as the **Pedal** or **Simson's Line** of the triangle ABC for the point P.

EXERCISES.

1. From any point P on the circum-circle of the triangle ABC, perpendiculars PD, PF are drawn to BC and AB: if FD, or FD produced, cuts AC at E, shew that PE is perpendicular to AC.

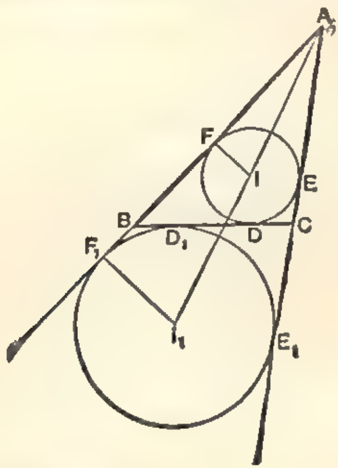
2. Find the locus of a point which moves so that if perpendiculars are drawn from it to the sides of a given triangle, their feet are collinear.

3. ABC and AB′C′ are two triangles with a common angle, and their circum-circles meet again at P; shew that the feet of perpendiculars drawn from P to the lines AB, AC, BC, B′C′ are collinear.

4. A triangle is inscribed in a circle, and any point P on the circumference is joined to the orthocentre of the triangle: shew that this joining line is bisected by the pedal of the point P.

THE TRIANGLE AND ITS CIRCLES.

VI. D, E, F *are the points of contact of the inscribed circle of the triangle* ABC, *and* D_1, E_1, F_1 *the points of contact of the escribed circle, which touches* BC *and the other sides produced:* a, b, c *denote the length of the sides* BC, CA, AB; s *the semi-perimeter of the triangle, and* r, r_1 *the radii of the inscribed and escribed circles.*



Prove the following equalities:

(i) $AE = AF = s - a,$
$BD = BF = s - b,$
$CD = CE = s - c.$

(ii) $AE_1 = AF_1 = s.$

(iii) $CD_1 = CE_1 = s - b,$
$BD_1 = BF_1 = s - c.$

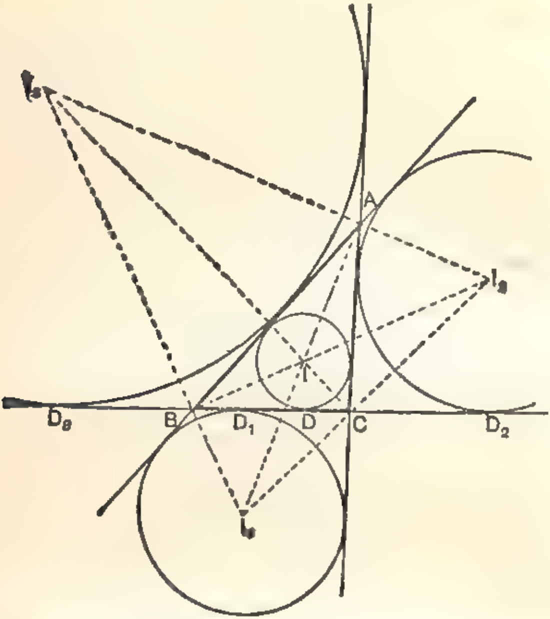
(iv) $CD = BD_1$, and $BD = CD.$

(v) $EE_1 = FF_1 = a.$

(vi) The area of the $\triangle ABC = rs$
$= r_1(s - a).$

(vii) Draw the above figure in the case when C is a right angle, and prove that
$$r = s - c; \quad r_1 = s - b.$$

VII. *In the triangle* ABC, I *is the centre of the inscribed circle, and* I_1, I_2, I_3 *the centres of the escribed circles touching respectively the sides* BC, CA, AB *and the other sides produced.*



Prove the following properties:

(i) *The points* A, I, I_1 *are collinear: so are* B, I, I_2; *and* C, I, I_3.

(ii) *The points* I_2, A, I_3 *are collinear; so are* I_3, B, I_1; *and* I_1, C, I_2.

(iii) *The triangles* BI_1C, CI_2A, AI_3B *are equiangular to one another.*

(iv) *The triangle* $I_1I_2I_3$ *is equiangular to the triangle formed by joining the points of contact of the inscribed circle.*

(v) *Of the four points* I, I_1, I_2, I_3, *each is the orthocentre of the triangle whose vertices are the other three.*

(vi) *The four circles, each of which passes through three of the points* I, I_1, I_2, I_3, *are all equal.*

EXERCISES.

1. With the figure given on page 214 shew that if the circles whose centres are I, I_1, I_2, I_3 touch BC at D, D_1, D_2, D_3, then

(i) $DD_2 = D_1D_3 = b$. (ii) $DD_3 = D_1D_2 = c$.
(iii) $D_2D_3 = b + c$. (iv) $DD_1 = b \sim c$.

2. *Shew that the orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.*

3. *Given the base and vertical angle of a triangle, find the locus of the centre of the escribed circle which touches the base.*

4. *Given the base and vertical angle of a triangle, shew that the centre of the circum-circle is fixed.*

5. Given the base BC, and the vertical angle A of the triangle, find the locus of the centre of the escribed circle which touches AC.

6. Given the base, the vertical angle, and the point of contact with the base of the in-circle; construct the triangle.

7. Given the base, the vertical angle, and the point of contact with the base, or base produced, of an escribed circle; construct the triangle.

8. I *is the centre of the circle inscribed in a triangle, and* I_1, I_2, I_3 *the centres of the escribed circles; shew that* II_1, II_2, II_3 *are bisected by the circumference of the circum-circle.*

9. ABC is a triangle, and I_2, I_3 the centres of the escribed circles which touch AC, and AB respectively: shew that the points B, C, I_2, I_3, lie upon a circle whose centre is on the circumference of the circum-circle of the triangle ABC.

10. With three given points as centres describe three circles touching one another two by two. How many solutions will there be?

11. Given the centres of the three escribed circles; construct the triangle.

12. Given the centre of the inscribed circle, and the centres of two escribed circles; construct the triangle.

13. Given the vertical angle, perimeter, and radius of the inscribed circle; construct the triangle.

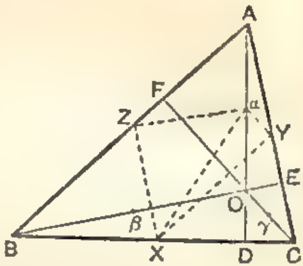
14. Given the vertical angle, the radius of the inscribed circle, and the length of the perpendicular from the vertex to the base; construct the triangle.

15. In a triangle ABC, I is the centre of the inscribed circle; shew that the centres of the circles circumscribed about the triangles BIC, CIA, AIB lie on the circumference of the circle circumscribed about the given triangle.

THE NINE-POINTS CIRCLE.

VIII. *In any triangle the middle points of the sides, the feet of the perpendiculars from the vertices to the opposite sides, and the middle points of the lines joining the orthocentre to the vertices are concyclic.*

In the $\triangle$ ABC, let X, Y, Z be the middle points of the sides BC, CA, AB; let D, E, F be the feet of the perp^s to these sides from A, B, C; let O be the orthocentre, and α, β, γ the middle points of OA, OB, OC.



It is required to prove that the nine points X, Y, Z, D, E, F, α, β, γ *are concyclic.*

Join XY, XZ, Xα, Yα, Zα.

Now from the $\triangle$ ABO, since AZ = ZB, and Aα = αO,
$\therefore$ Zα is par^l to BO. Ex. 2, p. 64.

And from the $\triangle$ ABC, since BZ = ZA, and BX = XC,
$\therefore$ ZX is par^l to AC.

But BO produced makes a rt. angle with AC;
$\therefore$ the $\angle$XZα is a rt. angle.

Similarly, the $\angle$XYα is a rt. angle.
$\therefore$ the points X, Z, α, Y are concyclic:
that is, α lies on the O^{ce} of the circle which passes through X, Y, Z; and Xα is a diameter of this circle.

Similarly it may be shewn that β and γ lie on the O^{ce} of this circle.

Again, since αDX is a rt. angle,
$\therefore$ the circle on Xα as diameter passes through D.

Similarly it may be shewn that E and F lie on the O^{ce} of this circle;
$\therefore$ the points X, Y, Z, D, E, F, α, β, γ are concyclic. Q.E.D.

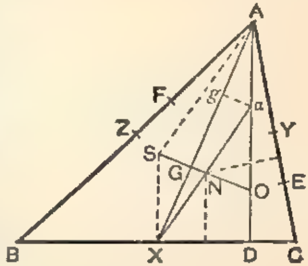
Obs. From this property the circle which passes through the middle points of the sides of a triangle is called the Nine-Points Circle; many of its properties may be derived from the fact of its being the circum-circle of the pedal triangle.

To prove that

(i) *the centre of the nine-points circle is the middle point of the straight line which joins the orthocentre to the circum-centre.*

(ii) *the radius of the nine-points circle is half the radius of the circum-circle.*

(iii) *the centroid is collinear with the circum-centre, the nine-points centre, and the orthocentre.*



In the $\triangle$ABC, let X, Y, Z be the middle points of the sides; D, E, F the feet of the perp^s; O the orthocentre; S and N the centres of the circumscribed and nine-points circles respectively.

(i) *To prove that* N *is the middle point of* SO.

It may be shewn that the perp. to XD from its middle point bisects SO; *Theor.* 22.

Similarly the perp. to EY at its middle point bisects SO:

that is, these perp^s intersect at the middle point of SO:

And since XD and EY are chords of the nine-points circle,

$\therefore$ the intersection of the lines which bisect XD and EY at rt. angles is its centre: *Theor.* 31, *Cor.* 1.

$\therefore$ the centre N is the middle point of SO. Q.E.D.

(ii) *To prove that the radius of the nine-points circle is half the radius of the circum-circle.*

By the last Proposition, Xα is a diameter of the nine-points circle.

$\therefore$ the middle point of Xα is its centre:

but the middle point of SO is also the centre of the nine-points circle. (*Proved.*)

Hence Xα and SO bisect one another at N.

Then from the $\triangle^s$ SNX, ONα,

$$\text{because}\begin{cases} SN = ON, \\ \text{and } NX = N\alpha, \\ \text{and the } \angle SNX = \text{the } \angle ON\alpha; \end{cases}$$

$$\therefore\ SX = O\alpha$$
$$= A\alpha.$$

And SX is also par^l to Aα,

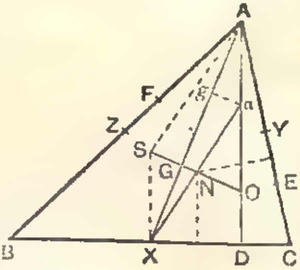
$\therefore$ SA = Xα.

But SA is a *radius* of the circum-circle;

and Xα is a *diameter* of the nine-points circle;

$\therefore$ the radius of the nine-points circle is half the radius of the circum-circle. [See also p. 267, Examples 2 and 3.] Q.E.D.

(iii) *To prove that the centroid is collinear with points* S, N, O.



Join AX and draw ag par[l] to SO.
Let AX meet SO at G.

Then from the $\triangle$AGO, since $Aa = aO$,
and ag is par[l] to OG,
$\therefore$ $Ag = gG$. Ex. 1, p. 64.

And from the $\triangle Xag$, since $aN = NX$,
and NG is par[l] to ag,
$\therefore$ $gG = GX$.
$\therefore$ $AG = \frac{2}{3}$ of AX;

$\therefore$ G is the centroid of the triangle ABC.
Theor. III., *Cor.*, p. 97.

That is, the centroid is collinear with the points S, N, O. Q.E.D.

EXERCISES.

1. *Given the base and vertical angle of a triangle, find the locus of the centre of the nine-points circle.*

2. The nine-points circle of any triangle ABC, whose orthocentre is O, is also the nine-points circle of each of the triangles AOB, BOC, COA.

3. If I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of a triangle ABC, then the circle circumscribed about ABC is the nine-points circle of each of the four triangles formed by joining three of the points I, I_1, I_2, I_3.

4. All triangles which have the same orthocentre and the same circumscribed circle, have also the same nine-points circle.

5. Given the base and vertical angle of a triangle, shew that one angle and one side of the pedal triangle are constant.

6. Given the base and vertical angle of a triangle, find the locus of the centre of the circle which passes through the three escribed centres.

NOTE. For some other important properties of the Nine-points Circle see Ex. 54, page 310.

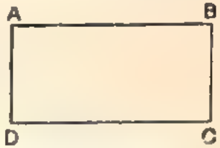
PART IV.

ON SQUARES AND RECTANGLES IN CONNECTION WITH THE SEGMENTS OF A STRAIGHT LINE.

THE GEOMETRICAL EQUIVALENTS OF CERTAIN ALGEBRAICAL FORMULÆ.

DEFINITIONS.

1. A rectangle ABCD is said to be contained by two adjacent sides AB, AD; for these sides fix its size and shape.



A rectangle whose adjacent sides are AB, AD is denoted by *the rect.* AB, AD; this is equivalent to the product AB AD.

Similarly a square drawn on the side AB is denoted by *the sq. on* AB, or AB^2.

2. If a point X is taken in a straight line AB, or in AB produced, then X is said to divide AB into the two segments AX, XB; the segments being in either case *the distances of the dividing point* X *from the extremities of the given line* AB.

Fig. 1.

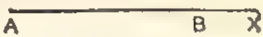


Fig. 2.

In Fig. 1, AB is said to be divided **internally** at X.

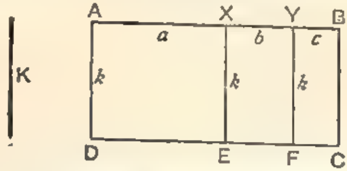
In Fig. 2, AB .... divided **externally** at X.

Obs. In internal division the given line AB is the *sum* of the segments AX, XB.

In external division the given line AB is the *difference* of the segments AX, XB.

THEOREM 50. [Euclid II. 1.]

If of two straight lines, one is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.



Let AB and K be the two given st. lines, and let AB be divided into any number of parts AX, XY, YB, which contain respectively a, b, and c units of length; so that AB contains $a+b+c$ units.

Let the line K contain k units of length.

It is required to prove that

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K;

namely that

$$(a+b+c)k = \quad ak \quad + \quad bk \quad + \quad ck.$$

Construction. Draw AD perp. to AB and equal to K.
Through D draw DC par[l] to AB.
Through X, Y, B draw XE, YF, BC par[l] to AD.

Proof. The fig. AC = the fig. AE + the fig. XF + the fig. YC; and of these, by construction,

fig. AC = rect. AB, K; and contains $(a+b+c)k$ units of area;

fig. AE = rect. AX, K; and contains ak units of area;
fig. XF = rect. XY, K; .... bk .... ;
fig. YC = rect. YB, K; .... ck ....

Hence

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K;

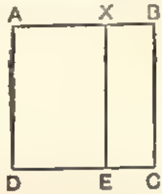
or, $$(a+b+c)k = \quad ak \quad + \quad bk \quad + \quad ck.$$

Q.E.D.

*COROLLARIES. [Euclid II. 2 and 3.]

Two special cases of this Theorem deserve attention.

(i) When AB is divided only at one point X, and when the undivided line AD is equal to AB.



Then the sq. on AB = the rect. AB, AX + the rect. AB, XB.

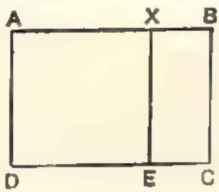
That is,

The square on the given line is equal to the sum of the rectangles contained by the whole line and each of the segments.

Or thus:

$$\begin{aligned} AB^2 &= AB \,.\, AB \\ &= AB(AX + XB) \\ &= AB \,.\, AX + AB \,.\, XB. \end{aligned}$$

(ii) When AB is divided at one point X, and when the undivided line AD is equal to one segment AX.



Then the rect. AB, AX = the sq. on AX + the rect. AX, XB.

That is,

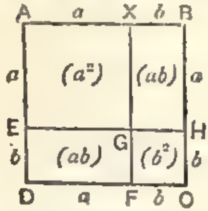
The rectangle contained by the whole line and one segment is equal to the square on that segment with the rectangle contained by the two segments.

Or thus:

$$\begin{aligned} AB \,.\, AX &= (AX + XB)AX \\ &= AX^2 + AX \,.\, XB. \end{aligned}$$

Theorem 51. [Euclid II. 4.]

If a straight line is divided internally *at any point, the square on the given line is equal to the sum of the squares on the two segments* together with *twice the rectangle contained by the segments.*



Let AB be the given st. line divided *internally* at X; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *sum* of the segments AX, XB, and therefore contains $a+b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 + 2AX \cdot XB;$$

namely that

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Construction. On AB describe a square ABCD. From AD cut off AE equal to AX, or a. Then $ED = XB = b$. Through E and X draw EH, XF par[l] respectively to AB, AD and meeting at G.

Proof. Then the fig. AC = the figs. AG, GC + the figs. EF, XH.

And of these, by construction,

fig. AC is the sq. on AB, and contains $(a+b)^2$ units of area;

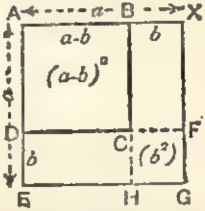
fig. AG = sq. on AX, and contains a^2 units of area;
fig. GC = sq. on XB, ... b^2 .... ;
fig. EF = rect. EG, ED
= rect. AX, XB .... ab .... ;
fig. XH = rect. GX, XB
= rect. AX, XB .... ab ....

Hence $$AB^2 = AX^2 + XB^2 + 2AX \cdot XB;$$
that is, $$(a+b)^2 = a^2 + b^2 + 2ab.$$

Q.E.D.

THEOREM 52. [Euclid II. 7.]

If a straight line is divided **externally** *at any point, the square on the given line is equal to the sum of the squares on the two segments* **diminished** *by twice the rectangle contained by the segments.*



Let AB be the given st. line divided *externally* at X; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *difference* of the segments AX, XB, and therefore contains $a - b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 - 2AX \cdot XB;$$

namely that $\quad (a-b)^2 = a^2 + b^2 - 2ab.$

Construction. On AX describe a square AXGE. From AE cut off AD equal to AB, or $a - b$. Then ED = XB = b. Through D and B draw DF, BH parˡ respectively to AX, AE, meeting at C.

Proof. Then the fig. AC = the figs. AG, CG – the figs. EF, XH.

And of these, by construction,

fig. AC is the sq. on AB, and contains $(a-b)^2$ units of area;

fig. AG = sq. on AX, and contains a^2 units of area;
fig. CG = sq. on XB, ... b^2 ....;
fig. EF = rect. EG, ED
= rect. AX, XB .... ab ....;
fig. XH = rect. GX, XB
= rect. AX, XB .... ab ....

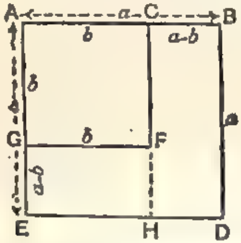
Hence $\quad AB^2 = AX^2 + XB^2 - 2AX \cdot XB;$

that is, $\quad (a-b)^2 = a^2 + b^2 - 2ab.$

Q.E.D.

THEOREM 53. [Euclid II. 5 and 6.]

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.



Let the given lines AB, AC be placed in the same st. line, and let them contain a and b units of length respectively.

It is required to prove that

$$AB^2 - AC^2 = (AB + AC)(AB - AC);$$

namely that $$a^2 - b^2 = (a+b)(a-b).$$

Construction. On AB and AC draw the squares ABDE, ACFG; and produce CF to meet ED at H.

Then $GE = CB = a - b$ units.

Proof. Now $AB^2 - AC^2$ = the sq. AD − the sq. AF

= the rect. CD + the rect. GH

$= DB . BC \quad + GF . GE$

$= AB . CB \quad + AC . CB$

$= (AB + AC)CB$

$= (AB + AC)(AB - AC).$

That is,

$$a^2 - b^2 = (a+b)(a-b).$$

Q.E.D

COROLLARY. *If a straight line is bisected, and also divided (internally or externally) into two unequal segments, the rectangle contained by these segments is equal to the difference of the squares on half the line and on the line between the points of section.*

Fig. 1.

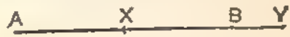


Fig. 2.

That is, if AB is bisected at X and also divided at Y, internally in Fig. 1, and externally in Fig. 2, then

in Fig. 1, $AY \cdot YB = AX^2 - XY^2$;

in Fig. 2, $AY \cdot YB = XY^2 - AX^2$.

For in the first case,
$$AY \cdot YB = (AX + XY)(XB - XY)$$
$$= (AX + XY)(AX - XY)$$
$$= AX^2 - XY^2.$$

The second case may be similarly proved.

EXERCISES.

1. Draw diagrams on squared paper to shew that the square on a straight line is
   (i) *four*-times the square on *half* the line;
   (ii) *nine*-times the square on *one-third* of the line.

2. Draw diagrams on squared paper to illustrate the following algebraical formulæ:
   (i) $(x+7)^2 = x^2 + 14x + 49$.
   (ii) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$.
   (iii) $(a+b)(c+d) = ac + ad + bc + bd$.
   (iv) $(x+7)(x+9) = x^2 + 16x + 63$.

3. In Theor. 50, Cor. (i), if AB = 4 cm., and the fig. AE = 9·6 sq. cm., find the area of the fig. XC.

4. In Theor. 50, Cor. (ii), if AX = 2·1", and the fig. XC = 3·36 sq. in., find AB.

5. In Theor. 51, if the fig. AG = 36 sq. cm., and the rect. AX, XB = 24 sq. cm., find AB.

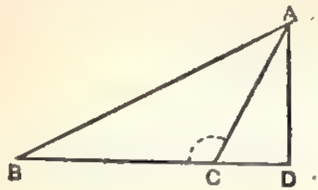
6. In Theorem 52, if the fig. AG = 9·61 sq. in., and the fig. DG = 6·51 sq. in., find AB.

[For further Examples on Theorems 50–53 see p. 230.]

✓

THEOREM 54. [Euclid II. 12.]

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with *twice the rectangle contained by one of those sides and the* projection of the other side *upon it.*



Let ABC be a triangle obtuse-angled at C; and let AD be drawn perp. to BC produced, so that CD is the projection of the side CA on BC. [See Def. p. 63.]

It is required to prove that

$$AB^2 = BC^2 + CA^2 + 2\,BC \,.\, CD.$$

Proof. Because BD is the *sum* of the lines BC, CD,

$$\therefore\ BD^2 = BC^2 + CD^2 + 2\,BC \,.\, CD. \qquad \textit{Theor. } 51.$$

To each of these equals add DA^2.

Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) + 2\,BC.\,CD.$

$$\left.\begin{array}{l}\text{But } BD^2 + DA^2 = AB^2\\ \text{and } CD^2 + DA^2 = CA^2\end{array}\right\}, \text{ for the } \angle D \text{ is a rt. } \angle.$$

Hence $$AB^2 = BC^2 + CA^2 + 2\,BC \,.\, CD.$$

Q.E.D.

Theorem 55. [Euclid II. 13.]

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle diminished by *twice the rectangle contained by one of those sides and the* projection of the other side *upon it.*

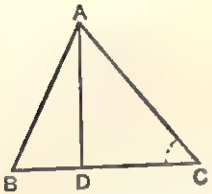


Fig. 1.

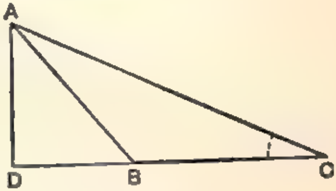


Fig. 2.

Let ABC be a triangle in which the $\angle$C is acute; and let AD be drawn perp. to BC, or BC produced; so that CD is the projection of the side CA on BC.

It is required to prove that

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Proof. Since in both figures BD is the *difference* of the lines BC, CD,

$$\therefore \; BD^2 = BC^2 + CD^2 - 2BC \cdot CD. \qquad \textit{Theor. } 52.$$

To each of these equals add DA^2.

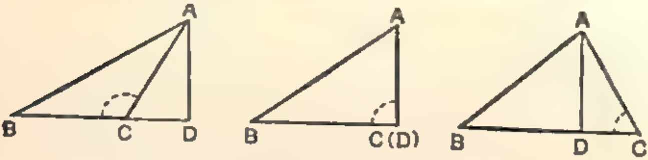
Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) - 2BC \cdot CD.$ ......(i)

But $\left.\begin{array}{l} BD^2 + DA^2 = AB^2 \\ \text{and } CD^2 + DA^2 = CA^2 \end{array}\right\}$, for the $\angle$D is a rt. $\angle$.

Hence $AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$

Q.E.D.

Summary of Theorems 29, 54 and 55.



(i) If the $\angle ACB$ is *obtuse*,

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$ *Theor.* 54

(ii) If the $\angle ACB$ is *a right angle*,

$$AB^2 = BC^2 + CA^2.$$ *Theor.* 29.

(iii) If the $\angle ACB$ is *acute*,

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$ *Theor.* 55.

Observe that in (ii), when the $\angle ACB$ is *right*, AD coincides with AC, so that CD (the projection of CA) vanishes;

hence, in this case, $2BC \cdot CD = 0$.

Thus the three results may be collected in a single enunciation:

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other sides, according as the angle contained by those sides is obtuse, a right angle, or acute; the difference in cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

EXERCISES.

1. In a triangle ABC, $a=21$ cm., $b=17$ cm., $c=10$ cm. By how many square centimetres does c^2 fall short of a^2+b^2? Hence or otherwise calculate the projection of AC on BC.

2. ABC is an isosceles triangle in which AB = AC; and BE is drawn perpendicular to AC. Shew that $BC^2 = 2AC \cdot CE$.

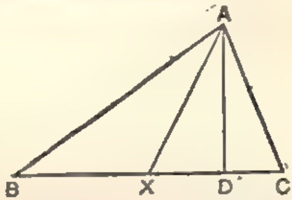
3. In the $\triangle ABC$, shew that

(i) if the $\angle C = 60°$, then $c^2 = a^2 + b^2 - ab$;

(ii) if the $\angle C = 120°$, then $c^2 = a^2 + b^2 + ab$.

THEOREM 56.

In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Let ABC be a triangle, and AX the median which bisects the base BC.

It is required to prove that

$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Draw AD perp. to BC; and consider the case in which AB and AC are unequal, and AD falls within the triangle.

Then of the $\angle^s$ AXB, AXC, one is obtuse, and the other acute Let the $\angle$ AXB be obtuse.

Then from the $\triangle$ AXB,

$$AB^2 = BX^2 + AX^2 + 2BX \cdot XD. \qquad \textit{Theor. 54.}$$

And from the $\triangle$ AXC,

$$AC^2 = XC^2 + AX^2 - 2XC \cdot XD. \qquad \textit{Theor. 55}$$

Adding these results, and remembering that XC = BX, we have

$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Q.E.D.

NOTE.—The proof may easily be adapted to the case in which the perpendicular AD falls outside the triangle.

EXERCISE.

In any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.

EXERCISES ON THEOREMS 50–53.

1. Use the Corollaries of Theorem 50 to shew that if a straight line AB is divided internally at X, then

$$AB^2 = AX^2 + XB^2 + 2AX \cdot XB.$$

2. If a straight line AB is bisected at X and produced to Y, and if $AY \cdot YB = 8AX^2$, shew that $AY = 2AB$.

3. *The sum of the squares on two straight lines is never less than twice the rectangle contained by the straight lines.*

Explain this statement by reference to the diagram of Theorem 52. Also deduce it from the formula $(a-b)^2 = a^2 + b^2 - 2ab$.

4. In the formula $(a+b)(a-b) = a^2 - b^2$, substitute $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$, and enunciate verbally the resulting theorem.

5. If a straight line is divided internally at Y, shew that the rectangle AY, YB continually diminishes as Y moves from X, the mid-point of AB.

Deduce this (i) from the Corollary of Theorem 53;

(ii) from the formula $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$.

6. *If a straight line* AB *is bisected at* X, *and also divided* (i) *internally,* (ii) *externally into two unequal segments at* Y, *shew that in either case*

$$AY^2 + YB^2 = 2(AX^2 + XY^2). \qquad \text{[Euclid II. 9, 10.]}$$

[*Proof of case* (i).

$$\begin{aligned} AY^2 + YB^2 &= AB^2 - 2AY \cdot YB && \textit{Theor. 51.} \\ &= 4AX^2 - 2(AX + XY)(AX - XY) \\ &= 4AX^2 - 2(AX^2 - XY^2) && \textit{Theor. 53.} \\ &= 2AX^2 + 2XY^2. \end{aligned}$$

Case (ii) may be derived from Theorem 52 in a similar way.]

7. If AB is divided internally at Y, use the result of the last example to trace the changes in the value of $AY^2 + YB^2$, as Y moves from A to B.

8. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.*

9. ABC is an isosceles triangle, and AY is drawn to cut the base BC internally or externally at Y. Prove that

$$AY^2 = AC^2 - BY \cdot YC, \text{ for internal section;}$$
$$AY^2 = AC^2 + BY \cdot YC, \text{ for external section.}$$

EXERCISES ON THEOREMS 54–56.

1. AB is a straight line 8 cm. in length, and from its middle point O as centre with radius 5 cm. a circle is drawn; if P is any point on the circumference, shew that

$$AP^2 + BP^2 = 82 \text{ sq. cm.}$$

2. In a triangle ABC, the base BC is bisected at X. If $a = 17$ cm., $b = 15$ cm., and $c = 8$ cm., calculate the length of the median AX, and deduce the $\angle A$.

3. The base of a triangle = 10 cm., and the sum of the squares on the other sides = 122 sq. cm.; find the locus of the vertex.

4. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

The sides of a rhombus and its shorter diagonal each measure 3"; find the longer diagonal to within ·01".

5. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [See Ex. 7, p. 64.]

6. ABCD is a rectangle, and O any point within it: shew that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

If AB = 6·0", BC = 2·5", and $OA^2 + OC^2 = 21\frac{1}{4}$ sq. in., find the distance of O from the intersection of the diagonals.

7. The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the straight line which joins the middle points of the diagonals.

8. In a triangle ABC, the angles at B and C are acute; if BE, CF are drawn perpendicular to AC, AB respectively, prove that

$$BC^2 = AB \cdot BF + AC \cdot CE.$$

9. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

10. ABC is a triangle, and O the point of intersection of its medians: shew that

$$AB^2 + BC^2 + CA^2 = 3(OA^2 + OB^2 + OC^2).$$

11. If a straight line AB is bisected at X, and also divided (internally or externally) at Y, then

$$AY^2 + YB^2 = 2(AX^2 + XY^2). \qquad \text{[See p. 230 Ex. 6.]}$$

Prove this from Theorem 56, by considering a triangle CAB in the limiting position when the vertex C falls at Y in the base AB.

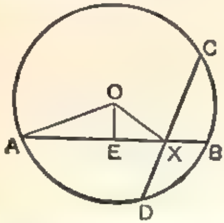
12. In a triangle ABC, if the base BC is divided at X so that mBX $= n$XC, shew that

$$mAB^2 + nAC^2 = mBX^2 + nXC^2 + (m+n)AX^2.$$

RECTANGLES IN CONNECTION WITH CIRCLES.

THEOREM 57. [Euclid III. 35.]

If two chords of a circle cut at a point within it, the rectangles contained by their segments are equal.



In the ⊙ ABC, let AB, CD be chords cutting at the internal point X.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD.

Let O be the centre, and r the radius, of the given circle.

Supposing OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX.

Proof. The rect. AX, XB $= (AE + EX)(EB - EX)$

$= (AE + EX)(AE - EX)$

$= AE^2 - EX^2$ *Theor.* 53.

$= (AE^2 + OE^2) - (EX^2 + OE^2)$

$= r^2 - OX^2$, since the ∠s at E are rt. ∠s.

Similarly it may be shewn that

the rect. CX, XD $= r^2 - OX^2$.

∴ the rect. AX, XB = the rect. CX, XD.

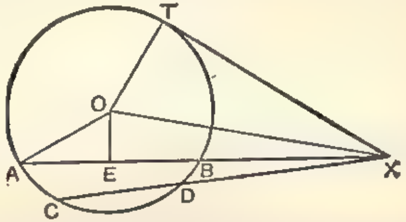
Q.E.D.

COROLLARY. *Each rectangle is equal to the square on half the chord which is bisected at the given point* X.

THEOREM 58. [Euclid III. 36.]

✓

If two chords of a circle, when produced, cut at a point outside it, the rectangles contained by their segments are equal. And each rectangle is equal to the square on the tangent from the point of intersection.



In the ⊙ABC, let AB, CD be chords cutting, when produced, at the external point X; and let XT be a tangent drawn from that point.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD = *the sq. on* XT.

Let O be the centre, and r the radius of the given circle.

Suppose OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX, OT.

Proof. The rect. AX, XB $= (EX + AE)(EX - EB)$

$= (EX + AE)(EX - AE)$

$= EX^2 - AE^2$ *Theor.* 53.

$= (EX^2 + OE^2) - (AE^2 + OE^2)$

$= OX^2 - r^2$, since

the $\angle$s at E are rt. $\angle$s.

Similarly it may be shewn that

the rect. CX, XD $= OX^2 - r^2$.

And since the radius OT is perp. to the tangent XT,

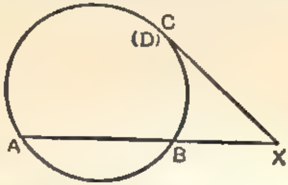
$\therefore XT^2 = OX^2 - r^2$. *Theor.* 29.

$\therefore$ the rect. AX, XB = the rect. CX, XD = the sq. on XT.

Q.E.D.

THEOREM 59. [Euclid III. 37.]

If from a point outside a circle two straight lines are drawn, one of which cuts the circle, and the other meets it; and if the rectangle contained by the whole line which cuts the circle and the part of it outside the circle is equal to the square on the line which meets the circle, then the line which meets the circle is a tangent to it.



From X a point outside the ⊙ABC, let two straight lines XA, XC be drawn, of which XA cuts the circle at A and B, and XC meets it at C;

and let the rect. XA, XB = the sq. on XC.

It is required to prove that XC *touches the circle at* C.

Proof. Suppose XC meets the circle again at D;

then $XA \cdot XB = XC \cdot XD$. *Theor.* 58.

But by hypothesis, $XA \cdot XB = XC^2$;

$\therefore\ XC \cdot XD = XC^2$;

$\therefore\ XD = XC$.

Hence XC cannot meet the circle again unless the points of section coincide;

that is, XC is a tangent to the circle.

Q.E.D.

NOTE ON THEOREMS 57, 58.

Remembering that the segments into which the chord AB is divided at X, internally in Theorem 57, and externally in Theorem 58, are in each case AX, XB, we may include both Theorems in a single enunciation.

If any number of chords of a circle are drawn through a given point within or without a circle, the rectangles contained by the segments of the chords are equal.

EXERCISES ON THEOREMS 57–59.

(Numerical and Graphical.)

1. Draw a circle of radius 5 cm., and within it take a point X 3 cm. from the centre O. Through X draw any two chords AB, CD.

(i) Measure the segments of AB and CD; hence find approximately the areas of the rectangles AX . XB and CX . XD, and compare the results.

(ii) Draw the chord MN which is bisected at X; and from the right-angled triangle OXM calculate the value of XM^2.

(iii) Find by how much per cent. your estimate of the rect. AX, XB differs from its true value.

2. Draw a circle of radius 3 cm., and take an external point X 5 cm. from the centre O. Through X draw any two secants XAB, XCD.

(i) Measure XA, XB and XC, XD; hence find approximately the rectangles XA . XB and XC . XD, and compare the results.

(ii) Draw the tangent XT; and from the right-angled triangle XTO calculate the value of XT^2.

(iii) Find by how much per cent. your estimate of the rect. AX, XB differs from its true value.

3. AB, CD are two straight lines intersecting at X. AX = 1·8", XB = 1·2", and CX = 2·7". If A, C, B, D are concyclic, find the length of XD.

Draw a circle through A, C, B, and check your result by measurement.

4. A secant XAB and a tangent XT are drawn to a circle from an external point X.

(i) If XA = 0·6", and XB = 2·4", find XT.

(ii) If XT = 7·5 cm., and XA = 4·5 cm., find XB.

5. A semi-circle is drawn on a given line AB; and from X, any point in AB, a perpendicular XM is drawn to AB cutting the circumference at M: shew that

$$AX . XB = MX^2.$$

(i) If AX = 2·5", and MX = 2·0", find XB; hence find the diameter of the semi-circle.

(ii) If the radius of the semi-circle = 3·7 cm., and AX = 4·9 cm., find MX.

6. A point X moves within a circle of radius 4 cm., and PQ is any chord passing through X; if in all positions PX . XQ = 12 sq. cm., find the locus of X.

What will the locus be if X moves outside the same circle, so that PX . XQ = 20 sq. cm.?

EXERCISES ON THEOREMS 57–59.

(*Theoretical.*)

1. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse: shew that

$$AD . DB = CD^2.$$

2. If two circles intersect, and through any point X in their common chord two chords AB, CD are drawn, one in each circle, shew that

$$AX . XB = CX . XD.$$

3. Deduce from Theorem 58 that the tangents drawn to a circle from any external point are equal.

4. If two circles intersect, tangents drawn to them from any point in their common chord produced are equal.

5. If a common tangent PQ is drawn to two circles which cut at A and B, shew that AB produced bisects PQ.

6. If two straight lines AB, CD intersect at X so that AX . XB = CX . XD, deduce from Theorem 57 (by *reductio ad absurdum*) that the points A, B, C, D are concyclic.

7. In the triangle ABC, perpendiculars AP, BQ are drawn from A and B to the opposite sides, and intersect at O: shew that

$$AO . OP = BO . OQ.$$

8. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse; shew that

$$AB . AD = AC^2.$$

9. Through A, a point of intersection of two circles, two straight lines CAE, DAF are drawn, each passing through a centre and terminated by the circumferences: shew that

$$CA . AE = DA . AF.$$

10. If from any external point P two tangents are drawn to a given circle whose centre is O and radius r; and if OP meets the chord of contact at Q; shew that

$$OP . OQ = r^2.$$

11. AB is a fixed diameter of a circle, and CD is perpendicular to AB (or AB produced); if *any* straight line is drawn from A to cut CD at P and the circle at Q, shew that

$$AP . AQ = \text{constant}.$$

12. A is a fixed point, and CD a fixed straight line; AP is *any* straight line drawn from A to meet CD at P; if in AP a point Q is taken so that AP . AQ is constant, find the locus of Q.

EXERCISES ON THEOREMS 57–59.

(*Miscellaneous.*)

1. The chord of an arc of a circle $=2c$, the height of the arc $=h$, the radius $=r$. Shew by Theorem 57 that

$$h(2r-h)=c^2.$$

Hence find the diameter of a circle in which a chord 24″ long cuts off a segment 8″ in height.

2. The radius of a circular arch is 25 feet, and its height is 18 feet: find the span of the arch.

If the height is reduced by 8 feet, the radius remaining the same, by how much will the span be reduced?

Check your calculated results graphically by a diagram in which 1″ represents 10 feet.

3. Employ the equation $h(2r-h)=c^2$ to find the height of an arc whose chord is 16 cm., and radius 17 cm.

Explain the double result geometrically.

4. If d denotes the shortest distance from an external point to a circle, and t the length of the tangent from the same point, shew by Theorem 58 that

$$d(d+2r)=t^2.$$

Hence find the diameter of the circle when $d=1{\cdot}2''$, and $t=2{\cdot}4''$; and verify your result graphically.

5. If the horizon visible to an observer on a cliff 330 feet above the sea-level is $22\frac{1}{4}$ miles distant, find roughly the diameter of the earth.

Hence find the approximate distance at which a bright light raised 66 feet above the sea is visible at the sea-level.

6. If h is the height of an arc of radius r, and b the chord of half the arc, prove that

$$b^2=2rh.$$

7. A semi-circle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P: shew that

$$AB^2=AC.AP+BD.BP.$$

8. Two circles intersect at B and C, and the two direct common tangents AE and DF are drawn: if the common chord is produced to meet the tangents at G and H, shew that

$$GH^2=AE^2+BC^2.$$

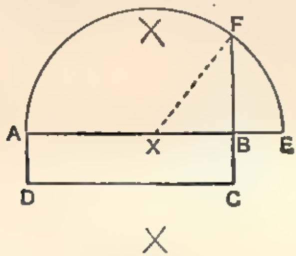
9. If from an external point P a secant PCD is drawn to a circle, and PM is perpendicular to a diameter AB, shew that

$$PM^2=PC.PD+AM.MB.$$

PROBLEMS.

PROBLEM 32.

To draw a square equal in area to a given rectangle.



Let ABCD be the given rectangle.

Construction. Produce AB to E, making BE equal to BC. On AE draw a semi-circle; and produce CB to meet the circumference at F.

Then BF is a side of the required square.

Proof. Let X be the mid-point of AE, and r the radius of the semi-circle. Join XF.

Then the rect. $AC = AB \cdot BE$
$= (r + XB)(r - XB)$
$= r^2 - XB^2$
$= FB^2$, from the rt. angled $\triangle$ FBX.

COROLLARY. *To describe a square equal in area to any given rectilineal figure.*

Reduce the given figure to a triangle of equal area. *Prob.* 18.
Draw a rectangle equivalent to this triangle. *Prob.* 17.
Apply to the rectangle the construction given above.

EXERCISES.

1. Draw a rectangle 8 cm. by 2 cm., and construct a square of equal area. What is the length of each side?

2. Find graphically the side of a square equal in area to a rectangle whose length and breadth are 3·0″ and 1·5″. Test your work by measurement and calculation.

3. Draw any rectangle whose area is 3·75 sq. in.; and construct a square of equal area. Find by measurement and calculation the length of each side.

4. Draw an equilateral triangle on a side of 3″, and construct a rectangle of equal area [Problem 17]. Hence find by construction and measurement the side of an equal square.

5. Draw a quadrilateral ABCD from the following data: $A=65°$; $AB=AD=9$ cm.; $BC=CD=5$ cm. Reduce this figure to a triangle [Problem 18], and hence to a rectangle of equal area. Construct an equal square, and measure the length of its side.

6. Divide AB, a line 9 cm. in length, internally at X, so that AX . XB = the square on a side of 4 cm.

Hence give a graphical solution, correct to the first decimal place, of the simultaneous equations:

$$x+y=9, \qquad xy=16.$$

7. Taking $\frac{1}{10}''$ as your unit of length, solve the following equations by a graphical construction, correct to one decimal figure:

$$x+y=40, \qquad xy=169.$$

8. The area of a rectangle is 25 sq. cm., and the length of one side is 7·2 cm.; find graphically the length of the other side to the nearest millimetre, and test your drawing by calculation.

9. Divide AB, a line 8 cm. in length, *externally* at X, so that AX . XB = the square on a side of 6 cm. [See p. 245.]

Hence find a graphical solution, correct to the first decimal place, of the equations:

$$x-y=8, \qquad xy=36.$$

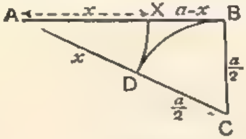
10. On a straight line AB draw a semi-circle, and from any point P on the circumference draw PX perpendicular to AB. Join AP, PB, and denote these lines by x and y.

Noticing that (i) $x^2+y^2=AB^2$; (ii) $xy=2\triangle APB=AB \, . \, PX$; devise a graphical solution of the equations:

$$x^2+y^2=100; \qquad xy=25.$$

PROBLEM 33.

To divide a given straight line so that the rectangle contained by the whole and one part may be equal to the square on the other part.



Let AB be the st. line to be divided at a point X in such a way that $AB \cdot BX = AX^2$.

Construction. Draw BC perp. to AB, and make BC equal to half AB. Join AC.

From CA cut off CD equal to CB.

From AB cut off AX equal to AD.

Then AB is divided as required at X.

Proof. Let $AB = a$ units of length, and let $AX = x$.

Then $BX = a - x$; $AD = x$; $BC = CD = \frac{a}{2}$.

Now $AB^2 = AC^2 - BC^2$, from the rt. angled $\triangle ABC$;

$= (AC - BC)(AC + BC)$;

that is, $a^2 = x(x + a)$

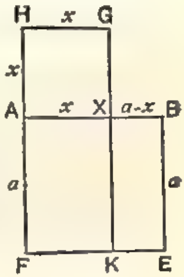
$= x^2 + ax$.

From each of these equals take ax;

then $a^2 - ax = x^2$;

or, $a(a - x) = x^2$,

that is, $AB \cdot BX = AX^2$.



EXERCISE.

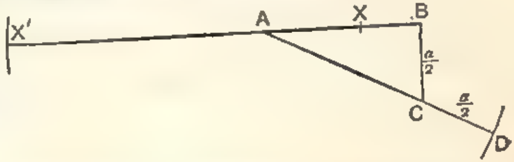
Let AB be divided as above at X. On AB, AX, and on opposite sides of AB, draw the squares ABEF, AXGH; and produce GX to meet FE at K. In this diagram name rectangular figures equivalent to a^2, x^2, $x(x+a)$, ax, and $a(a-x)$. Hence illustrate the above proof graphically.

NOTE. A straight line is said to be divided in **Medial Section** when the rectangle contained by the given line and one segment is equal to the square on the other segment.

This division may be *internal* or *external*; that is to say, AB may be divided internally at X, and externally at X′, so that

(i) $AB.BX = AX^2$,

(ii) $AB.BX' = AX'^2$.



To obtain X′, the construction of p. 240 must be modified thus:

CD is to be cut off from AC *produced*;

AX′.................. from BA *produced*, in the *negative* sense.

ALGEBRAICAL ILLUSTRATION.

If a st. line AB is divided at X, internally or externally, so that

$$AB.BX = AX^2,$$

and if $AB = a$, $AX = x$, and consequently $BX = a - x$, then

$$a(a-x) = x^2,$$

or,

$$x^2 + ax - a^2 = 0,$$

and the roots of this quadratic, namely, $\frac{a\sqrt{5}}{2} - \frac{a}{2}$ and $-\left(\frac{a\sqrt{5}}{2} + \frac{a}{2}\right)$, are the lengths of AX and AX′.

EXERCISES.

1. Divide a straight line 4″ long internally in medial section. Measure the greater segment, and find its length algebraically.

2. Divide AB, a line 2″ long, externally in medial section at X′. Measure AX′, and obtain its length algebraically, explaining the geometrical meaning of the negative sign.

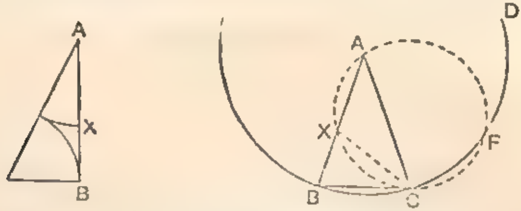
3. In the figure of Problem 33, shew that $AC = \frac{a\sqrt{5}}{2}$. [Theor. 29.]

Hence prove (i) $AX = \frac{a\sqrt{5}}{2} - \frac{a}{2}$; (ii) $AX' = -\left(\frac{a\sqrt{5}}{2} + \frac{a}{2}\right)$.

4. If a straight line is divided internally in medial section, and from the greater segment a part is taken equal to the less, shew that the greater segment is also divided in medial section.

Problem 34.

To draw an isosceles triangle having each of the angles at the base double of the vertical angle.



Construction. Take any line AB, and divide it at X, so that $AB \, . \, BX = AX^2$. *Prob.* 33.

(This construction is shewn separately on the left.)

With centre A, and radius AB, draw the $\odot$ BCD; and in it place the chord BC equal to AX.

Join AC.

Then ABC is the triangle required.

Proof. Join XC, and suppose a circle drawn through A, X and C.

Now, by construction, $BA \, . \, BX = AX^2$
$= BC^2$;

$\therefore$ BC touches the $\odot$ AXC at C; *Theor.* 59.

$\therefore$ the $\angle BCX =$ the $\angle XAC$, in the alt. segment.

To each add the $\angle XCA$;

then the $\angle BCA =$ the $\angle XAC +$ the $\angle XCA$
$=$ the ext. $\angle CXB$.

And the $\angle BCA =$ the $\angle CBA$, for $AB = AC$.

$\therefore$ the $\angle CBX =$ the $\angle CXB$;

$\therefore CX = CB = AX$,

$\therefore$ the $\angle XAC =$ the $\angle XCA$;

$\therefore$ the $\angle XAC +$ the $\angle XCA =$ twice the $\angle A$.

But the $\angle ABC =$ the $\angle ACB =$ the $\angle XAC +$ the $\angle XCA$ *Proved.*
$=$ twice the $\angle A$.

EXERCISES.

1. How many degrees are there in the vertical angle of an isosceles triangle in which each angle at the base is double of the vertical angle?

2. Shew how a right angle may be divided into five equal parts by means of Problem 34.

3. In the figure of Problem 34 point out a triangle whose vertical angle is three times either angle at the base.

Shew how such a triangle may be constructed.

4. If in the triangle ABC, the $\angle B$ = the $\angle C$ = twice the $\angle A$, shew that

$$\frac{BC}{AB} = \frac{\sqrt{5}-1}{2}.$$

5. In the figure of Problem 34, if the two circles intersect at F, shew that

(i) BC = CF;

(ii) the circle AXC = the circum-circle of the triangle ABC;

(iii) BC, CF are sides of a regular decagon inscribed in the circle BCD;

(iv) AX, XC, CF are sides of a regular pentagon inscribed in the circle AXC.

6. In the figure of Problem 34, shew that the centre of the circle circumscribed about the triangle CBX is the middle point of the arc XC.

7. In the figure of Problem 34, if I is the in-centre of the triangle ABC, and I′, S′ the in-centre and circum-centre of the triangle CBX, shew that S′I = S′I′.

8. If a straight line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

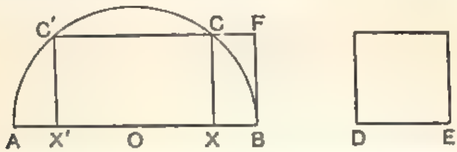
9. If a straight line AB is divided internally in medial section at X, shew that $$AB^2 + BX^2 = 3AX^2.$$

Also verify this result by substituting the values given on page 241.

THE GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS.

From the following constructions, which depend on Problem 32, a graphical solution of easy quadratic equations may be obtained.

I. *To divide a straight line* **internally** *so that the rectangle contained by the segments may be equal to a given square.*



Let AB be the st. line to be divided, and DE a side of the given square.

Construction. On AB draw a semicircle; and from B draw BF perp. to AB and equal to DE.

From F draw FCC′ par^l to AB, cutting the O^ce at C and C′.

From C, C′ draw CX, C′X′ perp. to AB.

Then AB is divided as required at X, and also at X′.

Proof. $AX . XB = CX^2$ *Prob.* 32.

$= BF^2$

$= DE^2$.

Similarly $AX' . X'B = DE^2$.

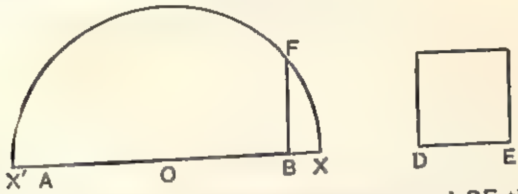
Application. The purpose of this construction is to find two straight lines AX, XB, having given their *sum*, viz. AB, and their *product*, viz. the square on DE.

Now to solve the equation $x^2 - 13x + 36 = 0$, we have to find two numbers whose *sum* is 13, and whose *product* is 36, or 6^2.

To do this graphically, perform the above construction, making AB equal to 13 cm., and DE equal to $\sqrt{36}$ or 6 cm. The segments AX, XB represent the roots of the equation, and their values may be obtained by measurement.

NOTE. If the last term of the equation is not a perfect square, as in $x^2 - 7x + 11 = 0$, $\sqrt{11}$ must be first got by the arithmetical rule, or graphically by means of Problem 32.

II. *To divide a straight line* **externally** *so that the rectangle contained by the segments may be equal to a given square.*



Let AB be the st. line to be divided externally, and DE the side of the given square.

Construction. From B draw BF perp. to AB, and equal to DE. Bisect AB at O.

With centre O, and radius OF draw a semi-circle to cut AB produced at X and X′.

Then AB is divided externally as required at X, and also at X′.

Proof. $$AX \cdot XB = X'B \cdot BX, \text{ since } AX = X'B,$$ *Prob.* 32

$$= BF^2$$

$$= DE^2.$$

Application. Here we find two lines AX, XB, having given their *difference*, viz. AB, and their *product*, viz. the square on DE.

Now to solve the equation $x^2-6x-16=0$, we have to find two numbers whose numerical *difference* is 6, and whose product is 16 or 4^2.

To do this graphically, perform the above construction, making AB equal to 6 cm., and DE equal to $\sqrt{16}$ or 4 cm. The segments AX, XB represent the roots of the equation, and their values, as before, may be obtained by measurement.

EXERCISES.

Obtain approximately the roots of the following quadratics by means of graphical constructions; and test your results algebraically.

1. $x^2-10x+16=0$.
2. $x^2-14x+49=0$.
3. $x^2-12x+25=0$.
4. $x^2-5x-36=0$.
5. $x^2-7x-49=0$.
6. $x^2-10x+20=0$.

EXERCISES FOR SQUARED PAPER.

1. A circle passing through the points (0, 4), (0, 9) touches the x-axis at P. Calculate and measure the length of OP.

2. With centre at the point (9, 6) a circle is drawn to touch the y-axis. Find the rectangle of the segments of any chord through O. Also find the rectangle of the segments of any chord through the point (9, 12).

3. Draw a circle (shewing all lines of construction) through the points (6, 0), (24, 0), (0, 9). Find the length of the other intercept on the y-axis, and verify by measurement. Also find the length of a tangent to the circle from the origin.

4. Draw a circle through the points (10, 0), (0, 5), (0, 20); and prove by Theorem 59 that it touches the x-axis.

Find (i) the coordinates of the centre, (ii) the length of the radius.

5. If a circle passes through the points (16, 0), (18, 0), (0, 12), shew by Theorem 58 that it also passes through (0, 24).

Find (i) the coordinates of the centre, (ii) the length of the tangent from the origin.

6. Plot the points A, B, C, D from the coordinates (12, 0), (-6, 0), (0, 9), (0, -8); and prove by Theorem 57 that they are concyclic.

If r denotes the radius of the circle, shew that

$$OA^2+OB^2+OC^2+OD^2=4r^2.$$

7. Draw a circle (shewing all lines of construction) to touch the y-axis at the point (0, 9), and to cut the x-axis at (3, 0).

Prove that the circle must cut the x-axis again at the point (27, 0); and find its radius. Verify your results by measurement.

8. Shew that two circles of radius 13 may be drawn through the point (0, 8) to touch the x-axis; and by means of Theorem 58 find the length of their common chord.

9. Given a circle of radius 15, the centre being at the origin, shew how to draw a second circle of the same radius touching the given circle and also touching the x-axis.

How many circles can be so drawn? Measure the coordinates of the centre of that in the first quadrant.

10. A, B, C, D are four points on the x-axis at distances 6, 9, 15, 25 from the origin O. Draw two intersecting circles, one through A, B, and the other through C, D, and hence determine a point P in the x-axis such that

$$PA \, . \, PB=PC \, . \, PD.$$

Calculate and measure OP.

If the distances of A, B, C, D from O are a, b, c, d respectively, prove that

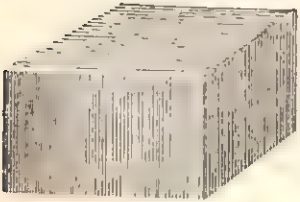
$$OP=(ab-cd)/(a+b-c-d).$$

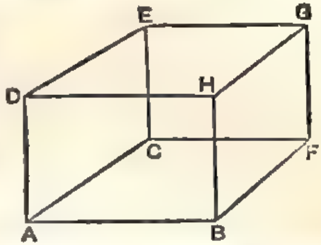
APPENDIX

APPENDIX.

ON THE FORM OF SOME SOLID FIGURES.

(*Rectangular Blocks.*)





The solid whose shape you are probably most familiar with is that represented by a brick or slab of hewn stone. This solid is called a **rectangular block** or **cuboid.** Let us examine its form more closely.

How many *faces* has it? How many *edges*? How many *corners*, or *vertices*?

The faces are quadrilaterals: of what shape?

Compare two opposite faces. Are they equal? Are they parallel?

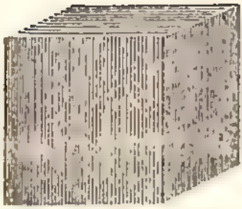
We may now sum up our observations thus:

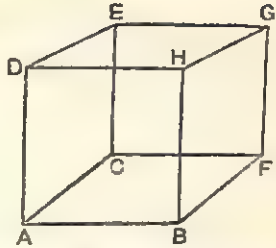
A cuboid has *six* faces; opposite faces being *equal rectangles in parallel planes.* It has *twelve* edges, which fall into three groups, corresponding to the *length*, the *breadth*, and the *height* of the block. The four edges in each group are equal and parallel, and perpendicular to the two faces which they cut.

The length, breadth, and height of a rectangular block are called its **three dimensions.**

Ex. 1. If two dimensions of a rectangular block are equal, say, the breadth AC and the height AD, two faces take a particular shape. Which faces? What shape?

Ex. 2. If the length, breadth, and height of a rectangular block are all equal, what shapes do the faces take?

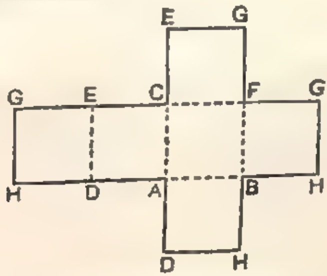




A rectangular block whose length, breadth, and height are all equal is called a cube. Its surface consists of six equal squares.

We will now see how models of these solids may be constructed, beginning with the cube, as being the simpler figure.

Suppose the surface of the cube to be cut along the upright edges, and also along the edge HG; and suppose the faces to be unfolded and flattened out on the plane of the base. The surface would then be represented by a figure consisting of six squares arranged as below.



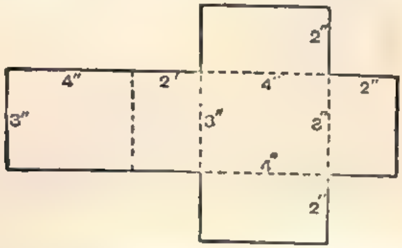
This figure is called the net of the cube: it is here drawn on half the scale of the cube shewn in outline above.

To make a model of a cube, draw its net on cardboard. Cut out the net along the outside lines, and cut partly through along the dotted lines. Fold the faces over till the edges come together; then fix the edges in position by strips of gummed paper.

Ex. 3. Make a model of a cube each of whose edges is 6·0 cm.

Ex. 4. Make a model of a rectangular block, whose length is 4", breadth 3", height 2".

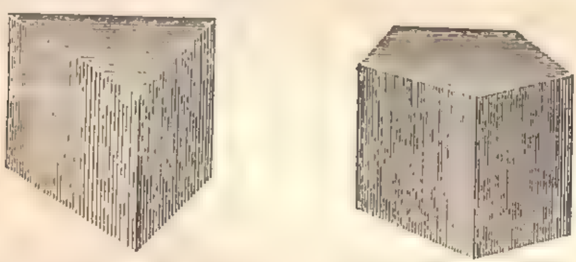
First draw the net which will consist of six rectangles arranged as below, and having the dimensions marked in the diagram.



Now cut the net out, fold the faces along the dotted lines, and secure the edges with gummed paper, as already explained.

(Prisms.)

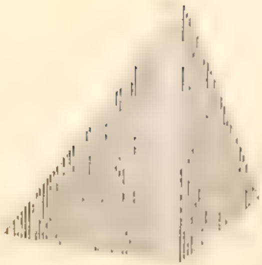
Let us now consider a solid whose side-faces (as in a rectangular block) are rectangles, but whose *ends* (*i.e.* base and top), though equal and parallel, are not necessarily *rectangles.* Such a solid is called a prism.

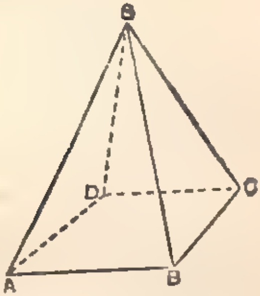


The ends of a prism may be any congruent figures; these may be triangles, quadrilaterals, or polygons of any number of sides. The diagram represents two prisms, one on a triangular base, the other on a pentagonal base.

Ex. 5. Draw the net of a triangular prism, whose ends are equilateral triangles on sides of 5 cm., and whose side-edges measure 7 cm.

(*Pyramids.*)





The solid represented in this diagram is called a pyramid.

The base of a pyramid (as of a prism) may have any number of sides, but the side-faces must be *triangles* whose vertices are at the same point.

The particular pyramid shewn in the Figure stands on a *square* base ABCD, and its side-edges SA, SB, SC, SD are all equal. In this case the side faces are equal isosceles triangles; and the pyramid is said to be *right*, for if the base is placed on a level table, then the vertex lies in an upright line through the mid-point of the base.

Ex. 6. Make a model of a right pyramid standing on a square base. Each edge of the base is to measure 3″, and each side-edge of the pyramid is to be 4″.

To make the necessary net, draw a square on a side of 3″. This will form the base of the pyramid. Then on the sides of this square draw isosceles triangles making the equal sides in each triangle 4″ long.

Explain why the process of folding about the dotted lines brings the four vertices together.

Another important form of pyramid has as base an equilateral triangle, and all the side edges are equal to the edges of the base.

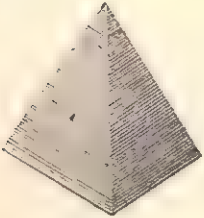


FIG. 1.

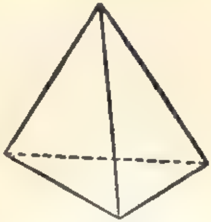


FIG. 2.

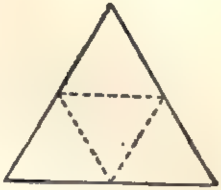


FIG. 3.

How many faces will such a pyramid have? How many edges? What sort of triangles will the side-faces be? Fig. 3 shews the net on a reduced scale.

A pyramid of this kind is called a regular tetrahedron (from Greek words meaning *four-faced*).

Ex. 7. Construct a model of a regular tetrahedron, each edge of which is 3″ long.

Ex. 8. What is the smallest number of *plane* faces that will enclose a space? What is the smallest number of *curved* surfaces that will enclose a space?

(*Cylinders.*)

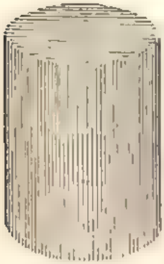


FIG. 1.

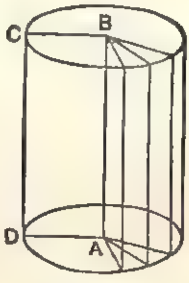


FIG. 2.

The solid figure here represented is called a cylinder.

On examining the model of which the last diagram is a drawing, you will notice that the two ends are *plane, circular, equal*, and *parallel*.

The side-surface is curved, but not curved in every direction; for it is evidently possible in one direction to rule *straight* lines on the surface: in *what* direction?

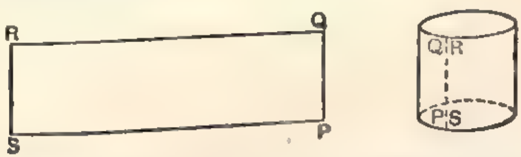
Let us take a rectangle ABCD (see Fig. 2), and suppose it to rotate about one side AB as a fixed axis.

What will BC and AD trace out, as they revolve about AB?

Observe that CD will move so as always to be parallel to the axis AB, and to pass round the curve traced out by D. As CD moves, it will generate (that is to say, *trace out*) a surface. What sort of surface?

We now see why in *one* direction, namely parallel to the axis AB, it is possible to rule *straight* lines on the *curved* surface of a cylinder.

It is easy to find a plane surface to represent the curved surface of a cylinder.



Cut a rectangular strip of paper, making the width PQ equal to the height of the cylinder. Wrap the paper round the cylinder, and carefully mark off the length PS that will make the paper go exactly once round. Cut off all that overlaps; and then unwrap the covering strip. You have now a rectangle representing the curved surface of the cylinder, and having the same area.

(*Cones.*)

FIG. 1.

FIG. 2.

We have now to examine the model of a cone, of which a drawing is given above.

Its surface consists of two parts; first a *plane circular* base, then a *curved surface* which tapers from the circumference of the base to a point above it called the vertex. Thus the form of a cone suggests a pyramid standing on a circular instead of a rectilineal base.

Let us take a triangle ABC right-angled at B (Fig. 2), and suppose it to rotate about one side AB as a fixed axis. What will BC trace out as the triangle revolves? Notice that AC will always pass through the *fixed* point A, and move round the curve traced out by C. As AC moves, it will generate a surface. What sort of surface?

We now see that the kind of cone represented in the diagram is a solid generated by the revolution of a right-angled triangle about one side containing the right angle.

Ex. 9. Why must the $\triangle$ ABC, rotating about AB, be *right-angled* at B, in order to generate a cone?

What would be generated by the revolution of an *obtuse*-angled triangle about one side forming the obtuse angle?

Ex. 10. What would be generated by an *oblique* parallelogram revolving about one side?

The curved surface of a cone may be represented by a plane figure thus:

Taking the slant-height AC of the cone as radius, draw a circle. Cut it out from your paper; call its centre A; and cut it along any radius AC. If you now place the centre of the circular paper at the vertex of the cone, you will find that you can wrap the paper round the cone without fold or crease. Mark off from the circumference of your paper the length CD that will go exactly once round the base of the cone; then cut through the radius AD. We have now a plane figure ACD (called a *sector of a circle*) which represents the curved surface of the cone, and has the same area.

(*Spheres.*)

The last solid we have to consider is the sphere, whose shape is that of a globe or billiard ball.

FIG. 1.

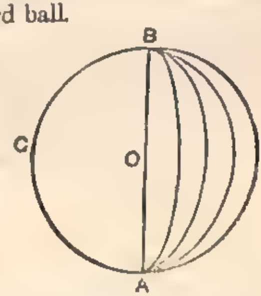


FIG. 2.

We shall realise its form more definitely, if we imagine a semi-circle ACB (Fig. 2) to rotate about its diameter as a fixed axis. Then, as the semi-circumference revolves, it generates the surface of a sphere.

Now since all points on the semi-circumference are in all positions at a constant distance from its centre O, we see that all points on the surface of a sphere are at a constant distance from a fixed point within it, namely the centre. This constant distance is the radius of the sphere. Thus all straight lines through the centre terminated both ways by the surface are equal: such lines are *diameters.*

Ex. 11. We have seen that on the curved surfaces of a cylinder and cone it is possible (in certain ways only) to rule *straight* lines. Is there any direction in which we can rule a straight line on the surface of a sphere?

Ex. 12. Again we have cut out a *plane* figure that could be wrapped round the *curved* surface of a cylinder without folding, creasing, or stretching. The same has been done for the curved surface of a cone. Can a flat piece of paper be wrapped about a sphere so as to fit all over the surface without creasing?

Ex. 13. Suppose you were to cut a sphere straight through the centre into two parts, in such a way that the new surfaces (made by cutting) are *plane,* these parts would be in every way alike. The parts into which a sphere is divided by a *plane central section* are called hemispheres. Of what shape is the line in which the plane surface meets the curved surface? If the section were *plane* but not *central,* can you tell what the meeting line of the two surfaces would be?

Ex. 14. If a cylinder were cut by a plane parallel to the base, of what shape would the new rim be?

Ex. 15. If a cone were cut by a plane parallel to the base, what would be the form of the section?

PART V.

ON PROPORTION.

Definitions and First Principles.

1. The **ratio** of one magnitude to another *of the same kind* is the relation which the first bears to the second in regard to quantity; this is measured by the fraction which the first is of the second.

Thus if two such magnitudes contain a and b units respectively, the ratio of the first to the second is expressed by the fraction $\frac{a}{b}$.

The ratio of a to b is generally denoted thus, $a : b$; and a is called the **antecedent** and b the **consequent** of the ratio.

The two magnitudes compared in a ratio must be *of the same kind*; for example, both must be lines, or both angles, or both areas. It is clearly impossible to compare the *length* of a straight line with a magnitude of a different kind, such as the *area* of a triangle. Moreover, a ratio is an *abstract* fraction. Thus the ratio which a line 6 cm. long bears to a line 8 cm. long is $\frac{6}{8}$ or $\frac{3}{4}$, (not $\frac{3}{4}$ *cm.*).

Note. It is not always possible to express two quantities of the same kind in terms of a common unit. For instance, if the side of a square is 1 inch, the diagonal is $\sqrt{2}$ inches. But since the numerical value of $\sqrt{2}$ cannot be *exactly* determined (though it can be found to any number of decimal figures), the side and diagonal cannot be expressed in terms of the same unit. Two such quantities are said to be **incommensurable**. But by choosing a sufficiently small quantity as unit, two incommensurables, such as $\sqrt{2}$ inches and 1 inch, may be expressed to any required degree of accuracy. Thus, remembering that $\sqrt{2} = 1 \cdot 41421 \ldots$, it follows that $\sqrt{2}$ inches and 1 inch may be represented by

1414 and 1000, roughly, taking $\frac{1}{1000}''$ as unit;

14142 and 10000, more nearly, taking $\frac{1}{10000}''$ as unit;

and so on.

2. If a point X is taken in a given line AB, or in AB produced, the ratio in which it divides AB is the ratio of the *segments* of AB, namely AX : XB, whether the division is *internal* as in Fig. 1, or *external* as in Fig. 2.

Fig. 1.

Fig. 2.

3. Four magnitudes are in **proportion**, when the ratio of the *first* to the *second* is equal to the ratio of the *third* to the *fourth*.

When the ratio a to b is equal to that of x to y, the four magnitudes are called **proportionals**.

This is expressed by saying "*a is to b as x is to y*": and the proportion is written

$$\frac{a}{b}=\frac{x}{y},$$

or

$$a:b=x:y.$$

Here a and y are called the **extremes**, and b and x the **means**; and y is said to be a **fourth proportional** to a, b, and x.

In a proportion, terms which are both antecedents or both consequents of the ratios are said to be **corresponding** terms.

NOTE. In a proportion such as $a:b=x:y$, the magnitudes compared in *each ratio* must be of the same kind, though the magnitudes of the second ratio need not be of the same kind as those of the first. For instance, a and b may denote *areas*, and x and y *lines*; in which case the proportion asserts that the first area bears the same ratio to the second area, as the first line bears to the second line.

4. Three magnitudes of the same kind are said to be proportionals, when the ratio of the *first* to the *second* is equal to that of the *second* to the *third*.

Thus a, b, c are proportionals if

$$\frac{a}{b}=\frac{b}{c};$$

or

$$a:b=b:c.$$

Here b is called a **mean proportional** between a and c;
and c is called a **third proportional** to a and b.

AXIOMS.

(i) *Ratios which are equal to the same ratio are equal to one another.*

For instance, if $a:b=x:y$, and $c:d=x:y$,
then evidently $a:b=c:d$.

(ii) *Magnitudes which bear the same ratio to the same magnitude are equal to one another.*

For instance, if $a : x = b : x$,

then evidently $a = b$.

INTRODUCTORY THEOREMS.

I. *If four magnitudes are proportionals, they are also proportionals when taken* inversely.

That is, if $a : b = x : y$,

then $b : a = y : x$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$; hence $\frac{b}{a} = \frac{y}{x}$;

or $b : a = y : x$.

II. *If four magnitudes of the same kind are proportionals, they are also proportionals when taken* alternately.

That is, if $a : b = x : y$,

then $a : x = b : y$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$;

multiplying both sides by $\frac{b}{x}$,

we have $\frac{a}{b} \cdot \frac{b}{x} = \frac{x}{y} \cdot \frac{b}{x}$;

that is, $\frac{a}{x} = \frac{b}{y}$,

or $a : x = b : y$.

NOTE. In this theorem the *hypothesis* requires that a and b shall be of the same kind, also that x and y shall be of the same kind; while the *conclusion* requires that a and x shall be of the same kind, and also b and y of the same kind.

III. *If four numbers are proportional, the* product of the extremes *is equal to the* product of the means.

That is, if $a : b = c : d$,

then $ad = bc$.

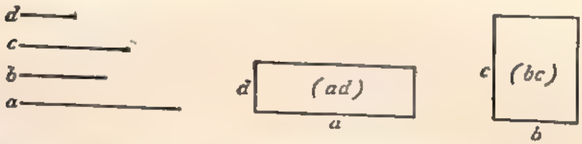
For, by hypothesis, $\frac{a}{b} = \frac{c}{d}$;

multiplying each side of this equation by bd, we have

$$ad = bc.$$

Corollary. If a, b, c, d denote the lengths of four straight lines in proportion, the above result states that *the* rectangle *contained by the extremes is equal to the* rectangle *contained by the means.*

This is illustrated by the following diagram :



Similarly if three lines, a, b, c are proportionals,

that is, if $a : b = b : c$;

then $ac = b^2$.

Or, *the* rectangle *contained by the extremes is equal to the* square *on the mean.*

IV. *If there are four magnitudes in proportion, the* sum (*or* difference) *of the first and second is to the second as the* sum (*or* difference) *of the third and fourth is to the fourth.*

That is, if $a : b = x : y$;

then (i) $a + b : b = x + y : y$;

(ii) $a - b : b = x - y : y$.

For, by hypothesis, $\frac{a}{b}=\frac{x}{y}$;

$$\therefore \frac{a}{b}+1=\frac{x}{y}+1, \text{ or } \frac{a+b}{b}=\frac{x+y}{y};$$

that is, $$a+b : b=x+y : y. \qquad \text{.........................(i)}$$

This inference is sometimes referred to as **componendo**.

Similarly by subtracting 1 from the equal ratios $\frac{a}{b}$, $\frac{x}{y}$, we obtain

$$\frac{a-b}{b}=\frac{x-y}{y};$$

that is, $$a-b : b=x-y : y. \qquad \text{.........................(ii)}$$

This inference is sometimes referred to as **dividendo**.

COROLLARY. If $a : b=x : y$,

then $$a+b : a-b=x+y : x-y.$$

This is obtained by dividing the result of (i) by that of (ii).

V. *In a series of equal ratios (the magnitudes being all of the same kind), as any antecedent is to its consequent so is the sum of the antecedents to the sum of the consequents.*

That is, if $$\frac{a}{x}=\frac{b}{y}=\frac{c}{z}=\ldots;$$

then $$\frac{a}{x}=\frac{a+b+c+\ldots}{x+y+z+\ldots}.$$

Let each of the equal ratios $\frac{a}{x}$, $\frac{b}{y}$, $\frac{c}{z}$, ... be equal to k.

Then $$a=kx,\ b=ky,\ c=kz,\ \ldots;$$

$\therefore$, by addition,

$$a+b+c+\ldots=k(x+y+z+\ldots);$$

$$\therefore \frac{a+b+c+\ldots}{x+y+z+\ldots}=k=\frac{a}{x},$$

or $$a : x=a+b+c+\ldots : x+y+z+\ldots.$$

VI. *A given straight line can be divided internally in a given ratio at one, and only one, point; and externally at one, and only one, point.*

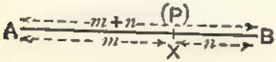


Fig. 1.

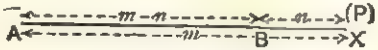


Fig. 2.

Let AB be the given line, and $m : n$ the given ratio, m being greater than n.

Internal Division. (i) Divide AB (Fig. 1) into $m+n$ equal parts [*Prob.* 7]; and of these parts make AX to contain m; then XB must contain n.

Hence $$AX : XB = m : n;$$

that is, AB is divided *internally* at X in the given ratio.

(ii) Again, since AX and AB contain respectively m and $m+n$ equal parts,

$$\therefore\ AX : AB = m : m+n.$$

Similarly, if P divides AB in the given ratio $m : n$,

$$AP : AB = m : m+n.$$

$$\therefore\ \frac{AX}{AB} = \frac{AP}{AB};$$

$$\therefore\ AX = AP.$$

Hence P and X coincide; that is, X is the only point which divides AB internally in the ratio $m : n$.

External Division. (i) Divide AB (Fig. 2) into $m-n$ equal parts; and in AB produced make AX to contain m such parts; then XB must contain n.

Hence $$AX : XB = m : n;$$

that is AB is divided *externally* at X in the given ratio.

(ii) And it may be shewn, as above, that X is the only point which divides AB externally in the ratio $m : n$.

EXERCISES.

1. Insert the missing terms in the following proportions:

(i) $3:7=15:(\quad)$;

(ii) $2\cdot5:(\quad)=10:32$;

(iii) $(\quad):ac^2=bc:bc^3$.

2. Correct the following statement:

£65 : 78 ft. = £25 : 30 ft.

3. If a straight line, 9·6″ in length, is divided *internally* in the ratio 5 : 7, calculate the lengths of the segments.

4. If a straight line, 4·5 cm. in length, is divided *externally* in the ratio 11 : 8, calculate the lengths of the segments.

5. AB is a straight line, 6·4 cm. in length, divided *internally* at X and *externally* at Y in the ratio 5 : 3; calculate the lengths of the segments, and shew that they satisfy the formula

$$\frac{2}{AB}=\frac{1}{AX}+\frac{1}{AY}.$$

6. If a straight line, a inches in length, is divided *internally* in the ratio $m:n$, shew that the lengths of the segments are respectively

$$\frac{m}{m+n}\cdot a \text{ inches}, \quad \frac{n}{m+n}\cdot a \text{ inches}.$$

7. If a straight line, a units in length, is divided *externally* in the ratio $m:n$, shew that the lengths of the segments are respectively

$$\frac{m}{m-n}\cdot a \text{ units}, \quad \frac{n}{m-n}\cdot a \text{ units}.$$

8. If $a:b=x:y$, and $b:c=y:z$, prove that $a:c=x:z$.

9. If $a:b=x:y$, shew that $a+b:a=x+y:x$.

10. If a, b, c are three proportionals, shew that $a:c=a^2:b^2$.

11. If two straight lines AB, CD are divided internally in the same ratio at X and Y respectively, shew that

(i) AB : XB = CD : YD;

(ii) AB : AX = CD : CY.

12. If a, b, c, d are four straight lines such that the rectangle contained by a and d is equal to that contained by b and c, prove that

$$a:b=c:d.$$

PROPORTIONAL DIVISION OF STRAIGHT LINES.

Theorem 60. [Euclid VI. 2.]

A straight line drawn parallel to one side of a triangle cuts the other two sides, or those sides produced, proportionally.

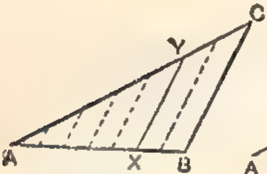


Fig. 1.

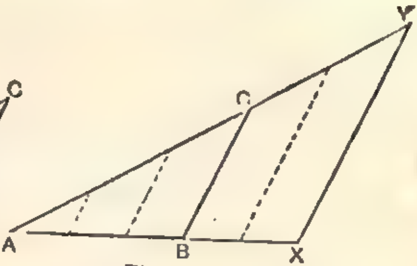


Fig. 2.

In the $\triangle$ABC, let XY, drawn par' to the side BC, cut AB, AC at X and Y, internally in Fig 1, externally in Fig 2.

It is required to prove in both cases that

$$AX : XB = AY : YC.$$

Proof. Suppose X divides AB in the ratio $m : n$; that is, suppose

$$AX : XB = m : n;$$

so that, if AX is divided into m equal parts, then XB may be divided into n such equal parts.

Through the points of division in AX, XB let parallels be drawn to BC.

Then these parallels divide the segments AY, YC into parts which are all equal; *Theor.* 22.

and of these equal parts AY contains m,

and YC contains n;

hence $AY : YC = m : n$.

$$\therefore AX : XB = AY : YC.$$

Q.E.D.

Conversely, *if a line cuts two sides of a triangle proportionally, it is parallel to the third side.*

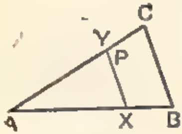


Fig. 1.

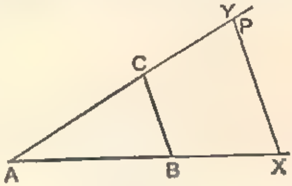


Fig. 2.

Conversely, let XY cut the sides AB, AC proportionally, so that

$$AX : XB = AY : YC.$$

It is required to prove that XY *is parallel to* BC.

Let XP be drawn through X par[1] to BC, to meet AC in P.

Then $AP : PC = AX : XB$;

but, by hypothesis, $AY : YC = AX : XB$.

Thus AC is cut, internally in Fig. 1, and externally in Fig. 2 in the same ratio at P and Y.

Hence P coincides with Y, and consequently XP with XY. *Theor.* VI. p. 252.

That is, XY is par[1] to BC. Q.E.D.

Corollary. If XY is parallel to BC, then

$$AX : AB = AY : AC.$$

For, taking Fig. 1, it may be shewn that

$$AX : AB = m : m + n;$$

and hence, by Theorem 22, that

$$AY : AC = m : m + n.$$

$$\therefore\ AX : AB = AY : AC.$$

Conversely, if $AX : AB = AY : AC$,

it may be proved as above that XY is par[1] to BC.

THEOREM 61. [Euclid VI. 3 and A.]

If the vertical angle of a triangle is bisected internally or externally, the bisector divides the base internally or externally into segments which have the same ratio as the other sides of the triangle.

Conversely, if the base is divided internally or externally into segments proportional to the other sides of the triangle, the line joining the point of section to the vertex bisects the vertical angle internally or externally.

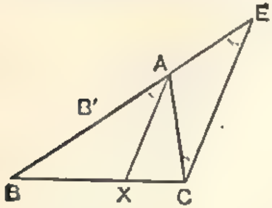


Fig. 1.

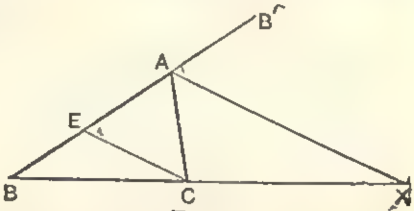


Fig. 2.

In the $\triangle$ABC, let AX bisect the $\angle$BAC, internally in Fig. 1, and externally in Fig. 2; that is, in the latter case, let AX bisect the exterior $\angle$B′AC.

It is required to prove in both cases that

$$BX : XC = BA : AC.$$

Let CE be drawn through C par[l] to XA to meet BA (produced, if necessary) at E. In Fig. 1 let a point B′ be taken in AB.

Proof. Because XA and CE are par[l],

∴, in both Figs., the $\angle$B′AX = the int. opp. $\angle$AEC.

Also, by hypothesis,

the $\angle$B′AX = the $\angle$XAC

= the alt. $\angle$ACE.

∴ the $\angle$AEC = the $\angle$ACE :

∴ AC = AE.

Again, because XA is par[l] to CE, a side of the $\triangle$BCE,

∴, in both Figs., BX : XC = BA : AE ;

that is, BX : XC = BA : AC.

Q.E.D.

Conversely, let BC be divided internally (Fig. 1) or externally (Fig. 2) at X, so that BX : XC = BA : AC.

It is required to prove that the $\angle$ B'AX = the $\angle$ XAC.

Proof. For, with the same construction as before,

because XA is par[l] to CE, a side of the $\triangle$BCE,

$\therefore$ BX : XC = BA : AE.

But, by hypothesis, BX : XC = BA : AC;

$\therefore$ BA : AC = BA : AE;

$\therefore$ AC = AE.

$\therefore$ the $\angle$ AEC = the $\angle$ ACE

= the alt. $\angle$ XAC.

And in both Figs.,

the ext. $\angle$ B'AX = the int. opp. $\angle$ AEC;

$\therefore$ the $\angle$ B'AX = the $\angle$ XAC.

Q.E.D.

Definition.

When a finite straight line is divided internally and externally into segments which have the same ratio, it is said to be **cut harmonically**.

Hence the following Corollary to Theorem 61.

The base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle:

for in each case the segments of the base are in the ratio of the other sides of the triangle.

[For Theorems and Examples on **Harmonic Section** see p. 323.]

EXERCISES ON THEOREM 60.

(Numerical and Graphical.)

1. On a base AB, 3·5″ in length, draw any triangle CAB; and from AB cut off AX 2·1″ long. Through X draw XY parallel to BC to meet AC at Y.

Measure AY, YC; and hence compare the ratios

$$\text{(i)}\ \frac{AX}{XB},\ \frac{AY}{YC};\quad \text{(ii)}\ \frac{AB}{AX},\ \frac{AC}{AY};\quad \text{(iii)}\ \frac{AB}{XB},\ \frac{AC}{YC}.$$

2. ABC is a triangle, and XY is drawn parallel to BC, cutting the other sides at X and Y.

(i) If AB = 3·6″, AC = 2·4″, and AX = 2·1″, calculate the length of AY.

(ii) If AB = 2·0″, AC = 1·5″, and AY = 0·9″, calculate the length of BX.

(iii) If X divides AB in the ratio 8 : 3, and if AC = 8·8 cm., find AY, YC.

3. ABC is a triangle, and XY is drawn parallel to BC, cutting the other sides produced at X and Y.

(i) If AB = 4·5 cm., AC = 3·5 cm., and AX = 7·2 cm., find by calculation and measurement the length of AY.

(ii) If X divides AB externally in the ratio 11 : 4, and if AC = 4·9 cm., find the segments of AC.

(Theoretical.)

4. *Three parallel straight lines cut any two transversals proportionally.*

5. The straight line which joins the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

6. Two triangles ABC, DBC stand on the same side of the common base BC; and from any point E in BC lines are drawn parallel to BA, BD, meeting AC, DC in F and G. Shew that FG is parallel to AD.

7. In a triangle ABC a transversal is drawn to cut the sides BC, CA, AB (produced if necessary) at D, E, and F respectively, and it makes equal angles with AB and AC; prove that

BD : CD = BF : CE.

EXERCISES ON THEOREM 61.

(Numerical and Graphical.)

1. Draw a triangle ABC, making $a=1{\cdot}5''$, $b=2{\cdot}4''$, and $c=3{\cdot}6''$. Bisect the angle A, internally and externally, by lines which meet BC and BC produced at X and Y.

Measure BX, XC; BY, YC; hence evaluate and compare the ratios

$$\frac{BX}{XC}, \frac{BY}{YC}, \frac{BA}{AC}.$$

2. In the triangle ABC, $a=3{\cdot}5$ cm., $b=5{\cdot}4$ cm., $c=7{\cdot}2$ cm.; and the internal and external bisectors of the $\angle$A meet BC at X and Y.

Calculate the lengths of the segments into which the base is divided at X and Y respectively; and verify your results graphically.

3. Frame constructions, based upon Theorem 61,

(i) to trisect a straight line of given length;

(ii) to divide a given line internally and externally in the ratio 3 : 2.

(Theoretical.)

4. AD is a median of the triangle ABC; and the angles ADB, ADC are bisected by lines which meet AB, AC at E and F respectively. Shew that EF is parallel to BC.

5. ABCD is a quadrilateral: shew that if the bisectors of the angles A and C meet in the diagonal BD, the bisectors of the angles B and D will meet on AC.

6. Employ Theorem 61 to shew that in any triangle

(i) the internal bisectors of the three angles are concurrent;

(ii) the external bisectors of two angles and the internal bisector of the third angle are concurrent.

7. If I is the in-centre of the triangle ABC, and if AI is produced to meet BC at X, shew that

$$AI : IX = AB + AC : BC.$$

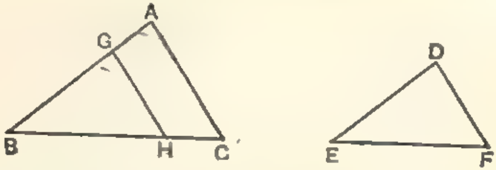
8. *Given the base of a triangle and the ratio of the other sides, find the locus of the vertex.*

9. Construct a triangle, having given the base, the ratio of the other sides, and the vertical angle.

EQUIANGULAR TRIANGLES.

Theorem 62. [Euclid VI. 4.]

If two triangles are equiangular to one another, their corresponding sides are proportional.



Let the $\triangle$s ABC, DEF have the $\angle$s A and B respectively equal to the $\angle$s D and E; and consequently the $\angle$C equal to the $\angle$F.

It is required to prove that

$$AB : DE = BC : EF = CA : FD.$$

Proof. Apply the $\triangle$DEF to the $\triangle$ABC, so that E falls on B, and EF along BC;

then since the $\angle$E = the $\angle$B, ED will fall along BA.

Let G and H be the points at which D and F fall respectively; so that GBH represents the $\triangle$DEF in its new position.

Now, by hypothesis, the $\angle$D = the $\angle$A;

that is, the ext. $\angle$BGH = the int. opp. $\angle$BAC;

$\therefore$ GH is par[l] to AC.

Hence BA : BG = BC : BH; *Theor. 60, Cor.*

that is, AB : DE = BC : EF.

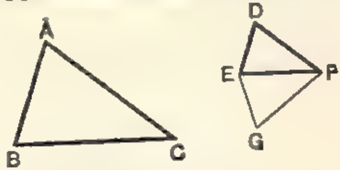
Similarly, by applying the $\triangle$DEF to the $\triangle$ABC, so that F falls on C, and FE, FD along CB, CA, it may be shewn that

$$BC : EF = CA : FD.$$

Hence finally, AB : DE = BC : EF = CA : FD. Q.E.D.

THEOREM 63. [Euclid VI. 5.]

If two triangles have their sides proportional when taken in order, the triangles are equiangular to one another, and those angles are equal which are opposite to corresponding sides.



In the $\triangle^s$ ABC, DEF, let

$$AB : DE = BC : EF = CA : FD.$$

It is required to prove that the $\triangle^s$ ABC, DEF *are equiangular to one another.*

At E in FE make the $\angle$ FEG equal to the $\angle$ B;

and at F in EF make the $\angle$ EFG equal to the $\angle$ C.

$\therefore$ the remaining $\angle$ EGF = the remaining $\angle$ A.

Proof. Since the $\triangle^s$ ABC, GEF are equiangular to one another,

$\therefore$ AB : GE = BC : EF. *Theor.* 62.

But, by hypothesis, AB : DE = BC : EF;

$\therefore$ AB : GE = AB : DE.

$\therefore$ GE = DE.

Similarly GF = DF.

Then in the $\triangle^s$ GEF, DEF,

because $\begin{cases} GE = DE, \\ GF = DF, \\ \text{and } EF \text{ is common;} \end{cases}$

$\therefore$ the triangles are identically equal; *Theor.* 7.

$\therefore$ the $\angle$ DEF = the $\angle$ GEF

= the $\angle$ B;

and the $\angle$ DFE = the $\angle$ GFE

= the $\angle$ C.

$\therefore$ the remaining $\angle$ D = the remaining $\angle$ A;

that is, the $\triangle$ DEF is equiangular to the $\triangle$ABC. Q.E.D.

EXERCISES ON EQUIANGULAR TRIANGLES.

(*Numerical and Graphical. The results are to be obtained by calculation and checked graphically.*)

1. In a triangle ABC, XY is drawn parallel to BC, cutting the other sides at X and Y:

(i) If AB = 2·5″, AC = 2·0″, AX = 1·5″; find AY.

(ii) If AB = 3·5″, AC = 2·1″, AY = 1·2″; find AX.

(iii) If AB = 4·2 cm., AX = 3·6 cm., AY = 6·6 cm.; find AC.

2. In the figure of the last example:

(i) If AB = 2·4″, BC = 3·6″, AX = 1·4″; find XY.

(ii) If BC = 7·7 cm., XY = 5·5 cm., AX = 4·5 cm.; find AB.

3. In the triangle ABC, $a = 3·0''$, $b = 3·6''$, $c = 4·2''$; and QR, drawn parallel to AC, measures 3·0″. Find the remaining sides of the triangle QBR.

4. ABC is a triangle in which $a = 8$ cm., $b = 7$ cm., and $c = 10$ cm. In AB a point P is taken 4 cm. from A, and PQ is drawn parallel to BC. Find the lengths of PQ and QC.

5. The sides of a triangular field are 400 yards, 350 yards, and 300 yards respectively. In a plan of the field the greatest side measures 2·4″; find the lengths of the other sides.

6. XY is drawn parallel to BC, the base of the triangle ABC. If AX = $8\frac{1}{3}$ ft., XY = $3\frac{1}{3}$ ft., AY = 6 ft. 2 in., and XB = $4\frac{1}{6}$ ft.; calculate the sides of the triangle ABC.

7. The triangle ABC is right-angled at C; and from P, a point in the hypotenuse, PQ is drawn parallel to AC.

If AC = $1\frac{1}{4}$″, BC = 3″, and PQ = $\frac{1}{2}$″; find BQ, BP, and AP.

8. In a triangle ABC, AD is the perpendicular from A on BC; and through X, a point in AD, a parallel is drawn to BC, meeting the other sides in P, Q.

If BC = 9 cm., AD = 8 cm., DX = 3 cm.; find PQ.

9. In the triangle ABC, $a = 2·0$ cm., $b = 3·5$ cm., $c = 4·5$ cm. BD and CE are drawn from the ends of the base to the opposite sides, and they intersect in P.

If
$$EP : PC = DP : PB = 2 : 5,$$
find the lengths of ED, AD, and DC.

EXERCISES ON EQUIANGULAR TRIANGLES.

(Theoretical.)

1. Shew that the straight line which joins the middle points of two sides of a triangle is

(i) parallel to the third side; (ii) one-half the third side.

2. In the trapezium ABCD, AB is parallel to DC, and the diagonals intersect at O: shew that

$$OA : OC = OB : OD.$$

If $AB = 2DC$, shew that O is a point of trisection on both diagonals.

3. If three concurrent straight lines are cut by two parallel transversals in A, B, C, and P, Q, R respectively; prove that

$$AB : BC = PQ : QR.$$

4. ABCD is a parallelogram, and from D a straight line is drawn to cut AB at E, and CB produced at F. In this figure name three triangles which are equiangular to one another; and shew that

$$DA : AE = FB : BE = FC : CD.$$

5. In the side AC of a triangle ABC any point D is taken: shew that if AD, DC, AB, BC are bisected in E, F, G, H respectively, then EG is equal to HF.

6. AB and CD are two parallel straight lines; E is the middle point of CD; AC and BE meet at F, and AE and BD meet at G: shew that FG is parallel to AB.

7. AB is a diameter of a circle, and through A any straight line is drawn to cut the circumference in C and the tangent at B in D: shew that

(i) the $\triangle^s$ CAB, BAD are equiangular to one another;

(ii) AC, AB, AD are three proportionals;

(iii) the rect. AC, AD is constant for all positions of AD.

8. If through any point X within a circle two chords AB, CD are drawn, and AC, BD joined; shew that

(i) the $\triangle^s$ AXC, DXB are equiangular to one another;

(ii) $AX : DX = XC : XB$.

Hence obtain an alternative proof of Theorem 57.

9. If from an external point X a tangent XT and a secant XAB are drawn to a circle, and AT, TB joined; shew that

(i) the $\triangle^s$ AXT, TXB are equiangular to one another;

(ii) $XA : XT = XT : XB$.

Hence obtain an alternative proof of Theorem 58.

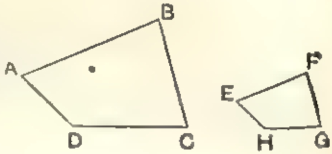
Definitions.

1. Two rectilineal figures are said to be **equiangular to one another** when the angles of the first, taken in order, are equal respectively to those of the second, taken in order.

2. Rectilineal figures are said to be **similar** when they are equiangular to one another, and also have their corresponding sides proportional.

Thus the two quadrilaterals ABCD, EFGH are similar if the angles at A, B, C, D are respectively equal to those at E, F, G, H, and if the following proportions hold:

$$\frac{AB}{EF}=\frac{BC}{FG}=\frac{CD}{GH}=\frac{DA}{HE}.$$



3. Similar figures are said to be **similarly described** with regard to two sides, when these sides correspond.

NOTE ON SIMILAR FIGURES.

Similar figures may be described as those which have the *same shape.* For this, *two* conditions are necessary:

(i) *the figures must have their angles equal each to each, taken in order*,

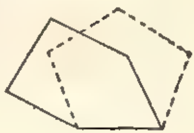
(ii) *their corresponding sides must be proportional.*

In the case of *triangles* we have learned that these conditions are not independent, for each follows from the other: thus

(i) if the triangles are *equiangular to one another*, Theorem 62 proves *that their corresponding sides are proportional*;

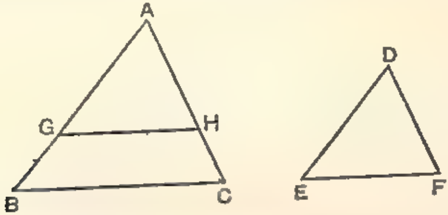
(ii) if the triangles have their *sides proportional*, Theorem 63 proves *that they are equiangular to one another.*

This, however, is not necessarily the case with rectilineal figures of more than three sides. For example, the first diagram in the margin shews two figures which are equiangular to one another, but which clearly have not their sides proportional; while the figures in the second diagram have their sides proportional, but are not equiangular to one another.



THEOREM 64. [Euclid VI. 6.]

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles are similar.



In the $\triangle^s$ ABC, DEF, let the $\angle$ A = the $\angle$ D,

and let AB : DE = AC : DF.

It is required to prove that the $\triangle^s$ ABC, DEF *are similar.*

Proof. Apply the $\triangle$ DEF to the $\triangle$ ABC, so that D falls on A, and DE along AB;

then because the $\angle$ EDF = the $\angle$ BAC, DF must fall along AC.

Let G and H be the points at which E and F fall respectively; so that AGH represents the $\triangle$ DEF in its new position.

Now, by hypothesis, AB : DE = AC : DF;

that is, AB : AG = AC : AH;

hence GH is par[l] to BC. *Theor.* 60, *Cor.*

$\therefore$ the ext. $\angle$ AGH, namely the $\angle$ E, = the int. opp. $\angle$ ABC;

and the ext. $\angle$ AHG, namely the $\angle$ F, = the int. opp. $\angle$ ACB.

Hence the $\triangle^s$ ABC, DEF are equiangular to one another,

so that their corresponding sides are proportional; *Theor.* 62.

that is, the $\triangle^s$ ABC, DEF are similar.

Q.E.D.

* THEOREM 65. [Euclid VI. 7.]

If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle in one proportional to the corresponding sides of the other, then the third angles are either equal or supplementary; and in the former case the triangles are similar.

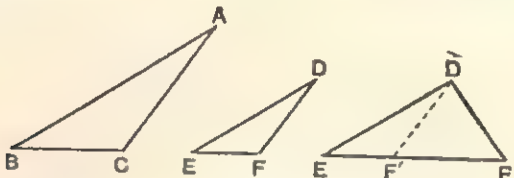


Fig. 1. Fig. 2. Fig. 3.

In the $\triangle^s$ ABC, DEF, let the $\angle$ B = the $\angle$ E; and let the sides about the $\angle^s$ A and D be proportional,

namely $$AB : DE = AC : DF.$$

It is required to prove that

either the $\angle$ C = the $\angle$ F, [as in Figs. 1 and 2];

or *the* $\angle$ C = *the supplement of the* $\angle$ F. [Figs. 1 and 3.]

Proof. (i) If the $\angle$ A = the $\angle$ D, [Figs. 1 and 2],

then the $\angle$ C = the $\angle$ F; *Theor.* 16.

and the $\triangle^s$ are equiangular, and therefore similar.

(ii) But if the $\angle$ A is not equal to the $\angle$ EDF [Figs. 1 and 3]

let the $\angle$ EDF′ = the $\angle$ A.

Then the $\triangle^s$ ABC, DEF′ are equiangular to one another;

$$\therefore\ AB : DE = AC : DF'.$$

But by hypothesis, $AB : DE = AC : DF$;

$$\therefore\ AC : DF' = AC : DF.$$

$$\therefore\ DF' = DF.$$

$\therefore$ the $\angle$ DFF′ = the $\angle$ DF′F.

But the $\angle$ C = the $\angle$ DF′E *Proved.*

= the supplement of the $\angle$ DF′F

= the supplement of the $\angle$ DFE.

Q.E.D.

EXERCISES ON SIMILAR TRIANGLES.

(*Theoretical.*)

1. In a triangle ABC, prove that any straight line parallel to the base BC and intercepted by the other two sides is bisected by the median drawn from the vertex A.

2. Two triangles ABC, A′B′C′ are equiangular to one another:

if p, p' denote the perpendiculars from A, A′ to the opp. sides,

R, R′............... circum-radii:

r, r' .... in-radii;

prove that each of the ratios $\frac{p}{p'}$, $\frac{R}{R'}$, $\frac{r}{r'}$ is equal to the ratio of any pair of corresponding sides.

3. Prove that the radius of the circle which passes through the mid-points of the sides of a triangle is half the circum-radius.

4. If two straight lines AB, CD intersect at X, so that

$$XA : XC = XD : XB;$$

(i) shew by Theorem 64 that the $\triangle$ˢ AXD, CXB are similar;
(ii) hence prove the points A, D, B, C concyclic.

5. A, B, C are three collinear points, and from B and C two parallel lines BP, CQ are drawn in the same sense, so that

$$PB : QC = AB : AC;$$

shew by Theorem 64 that the points, A, P, Q are collinear.

6. If in two triangles ABC, A′B′C′, the $\angle$ B = the $\angle$ B′, and $\frac{c}{c'} = \frac{b}{b'}$; what conclusion may be drawn?

Shew by diagrams how this conclusion is affected, if it is also given that
(i) c is less than b,
(ii) c is equal to b,
(iii) c is greater than b.

7. ABCD is a parallelogram; P and Q are points in a straight line parallel to AB; PA and QB meet at R, and PD and QC meet at S; shew that RS is parallel to AD.

8. In a triangle ABC the bisector of the vertical angle A meets the base at D and the circumference of the circum-circle at E; if EC is joined, shew that the triangles BAD, EAC are similar; and hence prove that

$$AB \cdot AC = AE \cdot AD.$$

Theorem 66. [Euclid VI. 8.]

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.

Let BAC be a triangle right-angled at A, and let AD be drawn perp. to BC.

It is required to prove that the $\triangle^s$ BDA, ADC *are similar to the* $\triangle$ BAC *and to one another.*

In the $\triangle^s$ BDA, BAC,
the $\angle$ BDA = the $\angle$ BAC, being rt. angles,
and the $\angle$ B is common to both ;
$\therefore$ the remaining $\angle$ BAD = the remaining $\angle$ BCA ; *Theor.* 16.
hence the $\triangle$ BDA is equiangular to the $\triangle$ BAC ;
$\therefore$ their corresponding sides are proportional ;
$\therefore$ the $\triangle^s$ BDA, BAC are similar.

In the same way it may be proved that the $\triangle^s$ ADC, BAC are similar.

Hence the $\triangle^s$ BDA, ADC, having their angles severally equal to those of the $\triangle$ BAC, are equiangular to one another ;

$\therefore$ they are similar.

Q.E.D.

Corollary. (i) Because the $\triangle^s$ DBA, DAC are similar,

$\therefore$ DB : DA = DA : DC ;

that is, DA is a mean proportional between DB and DC ;
and $DA^2 = DB \cdot DC$.

(ii) Because the $\triangle^s$ BCA, BAD are similar,

$\therefore$ BC : BA = BA : BD ;

hence $BA^2 = BC \cdot BD$.

(iii) Because the $\triangle^s$ CBA, CAD are similar,

$\therefore$ CB : CA = CA : CD ;

hence $CA^2 = CB \cdot CD$

EXERCISES.

(Miscellaneous Examples on Theorems 62-66.)

1. ABC is an equilateral triangle of which each side $=a$. In BC, produced both ways, two points P and Q are taken, such that $BP=CQ=a$, and AP, AQ are joined. Shew that

(i) $PQ : PA = PA : PB$.

(ii) $PA^2 = 3a^2$.

2. ABC is a triangle right-angled at A, and AD is drawn perpendicular to BC: if AB, AC measure respectively 4″ and 3″, shew that the segments of the hypotenuse are 3·2″ and 1·8″.

3. ABC is a triangle right-angled at A, and a perpendicular AD is drawn to the hypotenuse BC; shew (i) by Theorem 25, (ii) by Theorem 66, that

$$BC \cdot AD = AB \cdot AC.$$

4. ABC is a triangle right-angled at A, and AC′ is drawn perpendicular to the hypotenuse, also C′A′ is drawn parallel to CA. If $AC = 15$ cm., and $AB = 20$ cm., shew that $AC' = 12$ cm., and $C'A' = 9{\cdot}6$ cm.

5. At the extremities of a diameter of a circle, whose centre is C and radius r, tangents are drawn: these are cut in Q and R by any third tangent whose point of contact is P. Shew that

(i) QR subtends a right angle at C;

(ii) $PQ \cdot PR = r^2$.

6. Two circles of radii r and r' respectively have external contact at A, and a common tangent touches them at P and Q. Shew that

(i) PQ subtends a right angle at A; [Ex. 9. p. 187]

(ii) $PQ^2 = 4rr'$.

[Produce PA, QA to meet the circumferences at X and Y, and prove the triangles PAY, XAQ right-angled and similar.]

7. Two circles touch one another externally at A, and a common tangent PQ is produced to meet the line of centres at S. Shew that, if PA, AQ are joined,

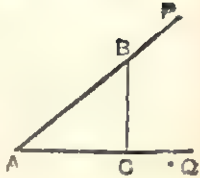
(i) the triangles SAP, SQA are similar;

(ii) $SA^2 = SP \cdot SQ$.

8. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D: shew that if BC, BD are joined,

$$BC : BA = BA : BD.$$

THE TRIGONOMETRICAL RATIOS.



1. Let PAQ be any acute angle: in AP, one of the arms of the angle, take a point B, and draw BC perp. to AQ.

Then with reference to the $\angle$A in the right-angled $\triangle$BAC, the following definitions are used.

The ratio $\frac{BC}{AB}$, or $\frac{\text{opposite side}}{\text{hypotenuse}}$, is called the sine of the $\angle$A.

The ratio $\frac{AC}{AB}$, or $\frac{\text{adjacent side}}{\text{hypotenuse}}$, ... cosine of the $\angle$A.

The ratio $\frac{BC}{AC}$, or $\frac{\text{opposite side}}{\text{adjacent side}}$, ... tangent of the $\angle$A.

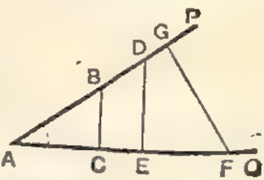
The reciprocals of these ratios are known respectively as the cosecant, the secant, and the cotangent of A.

These six ratios are called the trigonometrical ratios of the $\angle$A, and are usually expressed in the following shorter form.

$$\sin A = \frac{BC}{AB}, \quad \cos A = \frac{AC}{AB}, \quad \tan A = \frac{BC}{AC},$$

$$\operatorname{cosec} A = \frac{AB}{BC}, \quad \sec A = \frac{AB}{AC}, \quad \cot A = \frac{AC}{BC}.$$

NOTE. The squares of these ratios, namely $(\sin A)^2$, $(\cos A)^2$, ... are usually written in the form $\sin^2 A$, $\cos^2 A$, ...



2. In the adjoining figure, let BC, DE be perps. to AQ from points in AP, and let FG be perp. to AP from a point F in AQ.

Then the $\triangle$s BAC, DAE, FAG are similar, so that

$$\frac{BC}{AB} = \frac{DE}{AD} = \frac{FG}{AF}.$$

But these ratios express the value of sin A according as it is determined from the $\triangle$BAC, the $\triangle$DAE, or the $\triangle$FAG.

Thus sin A is unaltered so long as the $\angle$A remains the same. A similar proof holds for each of the trigonometrical ratios, shewing that they depend only on the size of the angle and not upon the lengths of its arms.

EXERCISES.

1. In a triangle ABC, right-angled at C, $a=8$, $b=15$; find c, and write down the values of sin A, cos A, and tan A.

2. In a right-angled triangle, the sides containing the right angle are 35 and 12: find the hypotenuse, and write down all the trigonometrical ratios of the smallest angle.

3. If A is any acute angle, shew that Theorem 29 may be made to assume either of the forms:

$$\text{(i)}\ \sin^2 A+\cos^2 A=1;\quad \text{(ii)}\ \sec^2 A=1+\tan^2 A.$$

4. ABCD is a quadrilateral in which the diagonal AC is at right angles to each of the sides AB, CD. If AB = 1·5 cm., AC = 3·6 cm., AD = 8·5 cm., draw the figure, and find sin ABC, tan ACB, cos CDA, tan DAC.

5. If A is any acute angle, shew that

$$\text{(i)}\ \sin(90^\circ-A)=\cos A;\quad \text{(ii)}\ \tan(90^\circ-A)=\cot A.$$

6. Construct an acute angle whose sine is 0·6. [See Prob. 10, p. 83.] Measure the angle with your protractor and give its value to the nearest degree.

7. Construct an acute angle A from each of the following data:

$$\text{(i)}\ \tan A=0{\cdot}7;\quad \text{(ii)}\ \cos A=0{\cdot}9;\quad \text{(iii)}\ \sin A=0{\cdot}71.$$

In each case measure the angle to the nearest degree.

8. Construct an acute angle A such that tan A = 1·6. Measure the angle A, and ascertain by measurement and by calculation the value of cos A.

9. Prove the following results

$$\text{(i)}\ \sin 45^\circ=\cos 45^\circ=\frac{1}{\sqrt{2}};\quad \text{(ii)}\ \sin 60^\circ=\cos 30^\circ=\frac{\sqrt{3}}{2}.$$

[See Ex. 11, p. 123, and Ex. 14, p. 124.]

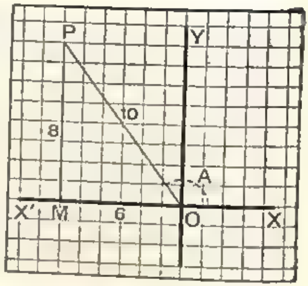
10. Construct a triangle ABC, right-angled at C, having the hypotenuse 10 cm. in length, and tan A = 0·81. Measure AC and the angle A; and find the values of sin A and cos A.

11. Draw a right-angled triangle ABC from the following data:

$$\tan A=0{\cdot}7,\quad \angle C=90^\circ,\quad b=2{\cdot}8\text{ cm.}$$

Measure c and the $\angle$A.

3. The definitions on page 270 may be extended to obtuse angles as follows:



Let XOX′ be a straight line, and let OY be perp. to it.

Let the angle A be traced by the revolution about O of the line OP which starts from the position OX.

Draw PM perp. to X′OX, thus forming a right-angled triangle POM. Then *whatever the position of* OP, the trigonometrical ratios of the angle A through which OP has turned are thus defined:

$$\sin A = \frac{PM}{OP}, \quad \cos A = \frac{OM}{OP}, \quad \tan A = \frac{PM}{OM},$$

with the understanding that OM *is to be considered positive when it is to the right of* OY, *and negative when to the left of* OY. [Compare p. 133.]

For example, in the above figure,

$$\sin A = \frac{PM}{OP} = \frac{8}{10} = \cdot 8.$$

$$\cos A = \frac{OM}{OP} = \frac{-6}{10} = -\cdot 6$$

$$\tan A = \frac{PM}{OM} = \frac{8}{-6} = -\frac{4}{3}.$$

Example. *To express trigonometrically*

(i) *the perpendicular from the vertex of a triangle on the base;*

(ii) *the projection of one side on another.*

(i) In both Figs., $\frac{AD}{AC} = \sin C$,

the sine of C being positive in each case.

$$\therefore p = b \sin C.$$

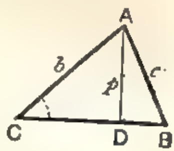


Fig. 1.

(ii) In Fig. 1, $\frac{CD}{CA} = \cos C$.

In Fig. 2, $\frac{CD}{CA}$ also represents $\cos C$,

if CD *is considered negative.*

∴ *numerically*

$$CD = +b \cos C \text{ in Fig. 1.}$$

$$CD = -b \cos C \text{ in Fig. 2.}$$

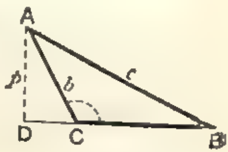


Fig. 2.

SOME GEOMETRICAL RESULTS EXPRESSED IN TRIGONOMETRICAL FORM.

[*The diagrams referred to are those of the preceding example.*]

1. In both Figs., $p = b \sin C$.

Similarly it may be proved that $p = c \sin B$.

Hence $b \sin C = c \sin B$; $\quad \therefore \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Similarly $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

that is, *the sides of a triangle are proportional to the sines of the opposite angles.*

2. From this property of a triangle deduce Theorem 62.

3. In both Figs.

$$\text{area of } \triangle ABC = \tfrac{1}{2}BC \,.\, AD = \tfrac{1}{2}ap;$$

and $$p = b \sin C; \quad \therefore \Delta = \tfrac{1}{2}ab \sin C.$$

Similarly $$\Delta = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$$

4. Express in trigonometrical form the area of

(i) a parallelogram, given two adjacent sides and the included angle.

(ii) a rhombus, given one side and one angle.

5. Shew that the circum-radius of a triangle is given by the formula
$$R = \frac{a}{2 \sin A} = \frac{abc}{4\Delta}.$$

6. In Fig. 1, we have
$$AB^2 = BC^2 + CA^2 - 2BC \,.\, CD. \qquad \textit{Theor. 55.}$$

In Fig 2, we have
$$AB^2 = BC^2 + CA^2 + 2BC \,.\, CD. \qquad \textit{Theor. 54.}$$

Now in Fig. 1, $\quad CD = +b \cos C$;

and in Fig. 2, $\quad CD = -b \cos C$.

Hence in both cases we have, on substitution,
$$c^2 = a^2 + b^2 - 2ab \cos C.$$

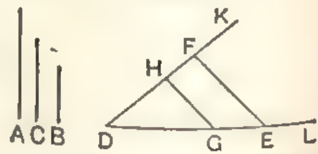
Similarly it may be shewn that
$$a^2 = b^2 + c^2 - 2bc \cos A.$$
$$b^2 = c^2 + a^2 - 2ca \cos B.$$

PROBLEMS.

PROBLEM 35.

To find the **fourth proportional** *to three given straight lines.*

Let A, B, C be the three given st. lines, to which the fourth proportional is required.



Construction. Draw two st. lines DL, DK of indefinite length, containing any angle.

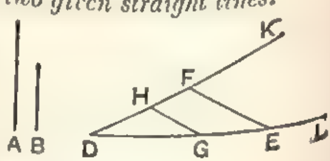
From DL cut off DG equal to A, and GE equal to B;
and from DK cut off DH equal to C.
Join GH.
Through E draw EF par[l] to GH.
Then HF is the fourth proportional to A, B, C.

Proof. Because GH is par[l] to EF, a side of the $\triangle$ DEF;
$\therefore$ DG : GE = DH : HF.
But DG = A, GE = B, and DH = C;
$\therefore$ A : B = C : HF;
that is, HF is the fourth proportional to A, B, C.

PROBLEM 36.

To find the **third proportional** *to two given straight lines.*

Let A, B be the two lines to which the third proportional is required.

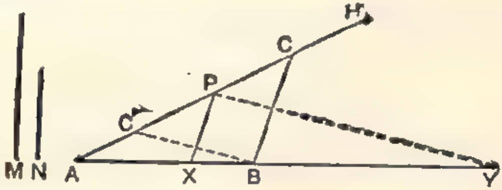


Construction. Draw two st. lines DL, DK.
From DL cut off DG equal to A, and GE equal to B;
and from DK cut off DH also equal to B.
Join GH.
Through E draw EF par[l] to GH.
Then HF is the third proportional to A, B.

Proof. As above, in Problem 35.

PROBLEM 37.

To divide a given straight line internally *and* externally *in a given ratio.*



Let AB be the st. line to be divided internally and externally in the ratio M : N.

Construction. From A draw a st. line AH at any angle with AB.

From AH cut off AP equal to M.

From PH and PA cut off PC and PC′, each equal to N.

Join BC, BC′.

Through P draw PX par[l] to BC, and PY par[l] to BC′.

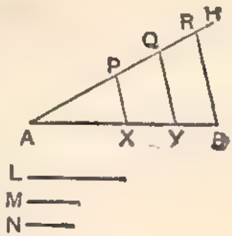
Then AB is divided internally at X, and externally at Y in the ratio M : N.

Proof. (i) Because PX is par[l] to BC, a side of the $\triangle$ABC,

$$\therefore \text{AX} : \text{XB} = \text{AP} : \text{PC}$$
$$= \text{M} : \text{N}.$$

(ii) Because PY is par[l] to BC′, a side of the $\triangle$ABC′,

$$\therefore \text{AY} : \text{YB} = \text{AP} : \text{PC}'$$
$$= \text{M} : \text{N}.$$



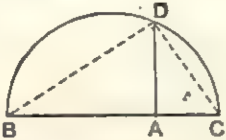
COROLLARY. By a similar process a st. line AB may be divided internally into segments proportional to three lines L, M, N.

Construction. Draw AH, and from it cut off AP, PQ, QR equal respectively to L, M, N. Join RB; and through P and Q draw PX, QY par[l] to BR.

Then evidently $\quad$ AX : L = XY : M = YB : N.

PROBLEM 38.

To find the **mean proportional** *between two given straight lines.*



Let AB, AC be the two given st. lines between which the mean proportional is to be found.

Construction. Place AB, AC in a straight line, and in opposite senses; and on BC describe the semi-circle BDC.

From A draw AD at rt. angles to BC, to cut the $\bigcirc^{ce}$ at D.

Then AD is the mean proportional between AB and AC.

Proof. Join BD, DC.

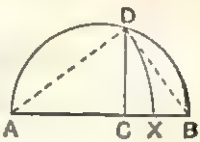
Now the $\angle$ BDC, being in a semi-circle, is a rt. angle.

And because in the right-angled $\triangle$ BDC, DA is drawn from the rt. angle perp. to the hypotenuse,

$\therefore$ the $\triangle^s$ ABD, ADC are similar; *Theor.* 66.

$\therefore$ AB : AD = AD : AC;

that is, AD is the mean proportional between AB and AC.



NOTE. If the given lines AB, AC are placed in the same sense, the mean proportional between them may be cut off from AB by the following useful construction.

On AB draw a semi-circle; and from C draw CD perp. to AB to cut the $\bigcirc^{ce}$ at D. From AB cut off AX equal to AD.

Then AX is the mean proportional between AB and AC.

For the $\triangle^s$ ABD, ADC are similar, *Theor.* 66.

$\therefore$ AB : AD = AD : AC;

that is, AB : AX = AX : AC.

GRAPHICAL EVALUATION OF A QUADRATIC SURD.

EXAMPLE. Find the approximate value of (i) $\sqrt{5}$, (ii) $\sqrt{21}$.

(i) $\sqrt{5}=\sqrt{5\times 1}$. Hence take AB, AC respectively to represent 5 and 1 in terms of any convenient unit, and find AD the mean proportional between them.

Then since
$$AB : AD = AD : AC,$$
$$\therefore\ AD^2 = AB \cdot AC$$
$$= 5\times 1 = 5.$$
$$\therefore\ AD = \sqrt{5}.$$

Thus *by measuring* AD, the value of $\sqrt{5}$ is roughly found to be 2·24.

(ii) $\sqrt{21}=\sqrt{7\times 3}$. Here take AB, AC equal to 7 cm. and 3 cm. respectively, and proceed as before.

NOTE. Factors should be chosen so as to give convenient lengths for AB, AC.

$$e.g.\ \sqrt{23}=\sqrt{2{\cdot}3\times 10};\quad \sqrt{11}=\sqrt{2{\cdot}2\times 5}.$$

DEFINITION.

A straight line is said to be divided in **extreme and mean ratio**, when the whole is to the greater segment as the greater segment is to the less.

Thus AB is divided at X in extreme and mean ratio,

when
$$AB : AX = AX : XB;$$

from which it follows that
$$AB \cdot BX = AX^2;$$

or, *the rectangle contained by the whole line and one part is equal to the square on the other part.*

Hence a straight line may be divided in extreme and mean ratio by Problem 33. For Construction and Proof see page 240.

EXERCISES.

1. Find graphically, testing your results by arithmetic;
   (i) The 4th proportional to 2·4″, 1·5″, 1·6″.
   (ii) The 3rd proportional to 2·5″ and 1·5″.
   (iii) The mean proportional between 7·2 cm. and 5·0 cm.

2. Divide a line, 2·0″ in length, internally and externally in the ratio 7 : 3; and in each case find the segments by measurement and calculation.

3. Obtain graphically the unknown term in the following statements of proportion; and check your result by arithmetic:
   (i) $1{\cdot}25 : x = 1{\cdot}0 : 1{\cdot}6$. [Take 1″ as the unit of length.]
   (ii) $x : 4{\cdot}2 = 4{\cdot}2 : 6{\cdot}3$. [Take 1 cm. as the unit of length.]
   (iii) $x : 16 = 25 : x$. [Let 1″ represent 10.]

4. Divide a line, 7·2 cm. in length, into three parts proportional to the numbers 2, 3, 4. Test your construction by measurement and calculation.

5. Divide a line, 3·9″ in length, into three parts, so that the second $= \frac{2}{3}$ of the first, and the third $= \frac{3}{4}$ of the second.

6. On a side of 1·5″ draw a rectangle equal in area to a square on a side of 2″. Measure the other side of the rectangle.

7. Find graphically the approximate values of

$$\text{(i) } \sqrt{3}; \quad \text{(ii) } \sqrt{10}; \quad \text{(iii) } \sqrt{\tfrac{14}{5}}.$$

8. Determine by geometrical constructions the approximate values of the following expressions, in each case verifying your drawing arithmetically:

$$\text{(i) } \frac{3{\cdot}5 \times 2{\cdot}4}{2{\cdot}8}; \quad \text{(ii) } \frac{6{\cdot}84}{2{\cdot}13}; \quad \text{(iii) } \frac{2{\cdot}71 \times 1{\cdot}26}{1{\cdot}51}.$$

9. Draw a triangle ABC from each of the following sets of data, and in each case calculate and measure the lengths of the sides:
   (i) The perimeter $= 4{\cdot}8''$; and $\frac{a}{3} = \frac{b}{4} = \frac{c}{5}$.
   (ii) The perimeter $= 11{\cdot}1$ cm.; and $a = \frac{5}{6}b$, $b = \frac{4}{5}a$.
   (iii) The perimeter $= 11{\cdot}8$ cm.; and $\frac{A}{1} = \frac{B}{2} = \frac{C}{3}$.
   (iv) $a = 4{\cdot}0''$, $A = 90°$; and $b : c = 5 : 3$.

EXERCISES.

(Proportion applied to the calculation of Heights and Distances.)

1. A field is represented in a plan by a triangle ABC, in which $a=8$ cm., $b=5{\cdot}6$ cm., $c=6{\cdot}4$ cm. If the greatest side of the field is 200 metres, find the lengths of the other sides.

A fence, run across the field, is represented in the plan by a line PQ parallel to BC drawn from a point P in AB distant 4·0 cm. from A. Find the length of the fence.

2. A's speed is to B's in the ratio 8 : 7; find *graphically* by how much A would beat B in a 100 yards' race, supposing each man to run throughout at a uniform rate.

3. On a map in which 1″ represents 25 miles, three places, A, B, and C are marked. Of these, B appears N.W. of A at a distance 0·8″; and C appears N.E. of A at a distance 1·5″. Find the actual distance between B and C.

4. A man, whose height is 6 feet, standing 32 feet from a lamp-post, observes that his shadow cast by the light is 8 feet in length: how high is the light above the ground, and how long would be the shadow of a boy 5 feet in height standing 20 feet from the post?

5. A man 6 feet in height, standing 15 feet from a lamp-post, observes that his shadow cast by the light is 5 feet in length: how high is the light, and how long would his shadow be if he were to approach 8 feet nearer to the post?

6. To find the width of a canal a rod is fixed vertically on the bank so as to shew $4\frac{1}{4}$ feet of its length. The observer, whose eye is 5 ft. 8 in above the ground retires at right angles from the canal until he sees the top of the rod in a line with the further bank. If his distance from the nearer bank is now 20 feet, what is the width of the canal?

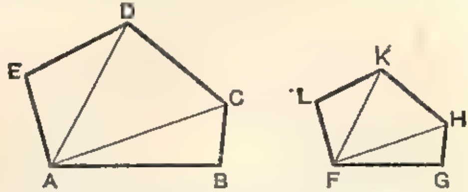
7. A man, wishing to ascertain the height of a tower, fixes a staff vertically in the ground at a distance of 27 ft. from the tower. Then, retiring 3 ft. farther from the tower, he sees the top of the staff in line with the top of the tower. If the observer's eye and the top of the staff are respectively 5 ft. 4 in. and 12 ft. above the ground, find the height of the tower.

8. A person due S. of a lighthouse observes that his shadow cast by the light at the top is 24 feet long. On walking 100 yards due E. he finds his shadow to be 30 feet long. Supposing him to be 6 feet high, find the height of the light from the ground.

SIMILAR FIGURES.

THEOREM 67.

Similar polygons can be divided into the same number of similar triangles; and the lines joining corresponding vertices in each figure are proportional.



Let ABCDE, FGHKL be similar polygons, the vertex A corresponding to the vertex F, B to G, and so on. Let AC, AD be joined, and also FH, FK.

It is required to prove that

(i) the $\triangle^s$ ABC, FGH *are similar; as also the* $\triangle^s$ ACD, FHK, *and the* $\triangle^s$ ADE, FKL.

(ii) $$AB : FG = AC : FH = AD : FK.$$

Proof. (i) Since the polygons are similar,

the $\angle ABC =$ the $\angle FGH$,

and $AB : FG = BC : GH$;

$\therefore$ the $\triangle^s$ ABC, FGH are similar. *Theor.* 64.

$\therefore$ the $\angle BCA =$ the $\angle GHF$;

but because the polygons are similar,

the $\angle BCD =$ the $\angle GHK$;

$\therefore$ the $\angle ACD =$ the $\angle FHK$.

Also $AC : FH = BC : GH$, for the $\triangle^s$ ABC, FGH are similar,

$= CD : HK$, for the polygons are similar.

That is, the sides about the equal $\angle^s$ ACD, FHK are proportional,

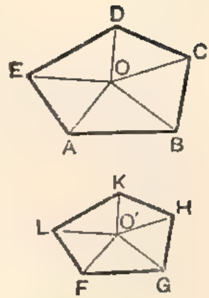
$\therefore$ the $\triangle^s$ ACD, FHK are similar. *Theor.* 64.

In the same way the $\triangle^s$ ADE, FKL are similar.

(ii) And $AB : FG = AC : FH$, from the similar $\triangle^s$ ABC, FGH;

$= AD : FK$, from the similar $\triangle^s$ CAD, HFK.

Q.E.D.



NOTE. In the last Theorem the polygons have been divided into similar triangles by lines drawn from a pair of *corresponding vertices.* But this restriction is not necessary.

For take *any* point O in the polygon ABCDE, and join it to each of the vertices.

In the polygon FGHKL, make the ∠ GFO′ equal to the ∠ BAO,

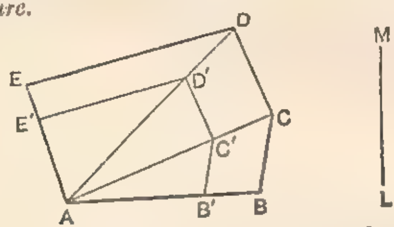
and make the ∠ FGO′ equal to the ∠ ABO.

Join O′ to each vertex of the polygon FGHKL.

We leave as an exercise to the student the proof that the two polygons are thus divided into the same number of similar triangles.

PROBLEM 39. [First Method.]

On a side of given length to draw a figure similar to a given rectilineal figure.



Let ABCDE be the given figure, and LM the length of the given side; and suppose that this side is to correspond to AB.

Construction. From AB cut off AB′ equal to LM.

Join AC, AD.

From B′ draw B′C′ par[l] to BC, to cut AC at C′.

From C′ draw C′D′ par[l] to CD, to cut AD at D′.

From D′ draw D′E′ par[l] to DE, to cut EA at E′.

Then AB′C′D′E′ is the required figure.

Outline of Proof. (i) By construction the figure AB′C′D′E′ is equiangular to the figure ABCDE.

(ii) From the three pairs of similar triangles it may be shewn

that $$\frac{AB'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'E'}{DE} = \frac{E'A}{EA};$$

that is, corresponding sides of the polygons are proportional.

Theorem 68.

Any two similar rectilineal figures may be so placed that the lines joining corresponding vertices are concurrent.

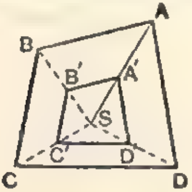


Fig. 1.

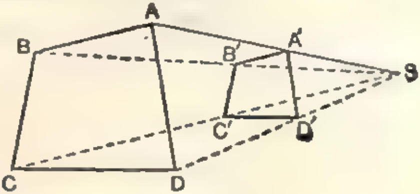


Fig. 2.

Let ABCD, A′B′C′D′ be similar figures.

Then since the ∠B′ = the ∠B, the figures can be so placed that A′B′, B′C′ are respectively par[l] to AB, BC. It follows, since the figures are equiangular to one another, that C′D′ is par[l] to CD, and D′A′ par[l] to DA.

It is required to prove that when corresponding sides of the given figures are parallel, then AA′, BB′, CC′, DD′ *are concurrent.*

Join AA′, and divide it externally at S in the ratio AB : A′B′.
Join SB *and* SB′ : it will be shewn that SB and SB′ are in one straight line.

Proof. In the △[s] SAB, SA′B′, since AB and A′B′ are par[l],

∴ the ∠SAB = the ∠SA′B′ ;

and, by construction, SA : SA′ = AB : A′B′ ;

∴ the △[s] SAB, SA′B′ are equiangular to one another ; *Theor.* 64.

∴ the ∠ASB = the ∠A′SB′.

Hence SB, SB′ are in the same st. line ;

that is, BB′ passes through the fixed point S.

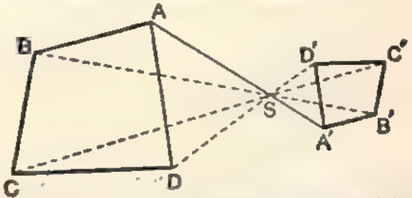
Similarly CC′ and DD′ may be shewn to pass through S.

That is, AA′, BB′, CC′, DD′ are concurrent. Q.E.D.

Note. Observe that the joining lines AA′, BB′, CC′, DD′ are all divided externally at S in the ratio of any pair of corresponding sides of the given figures.

Note. In placing the given figures so that A′B′, B′C′ are respectively parallel to AB, BC, two cases arise :

(i) A′B′ and AB may have *the same sense*, as in Figs. 1 and 2 ;
(ii) A′B′ and AB .... *opposite senses*, as in the Fig. below.

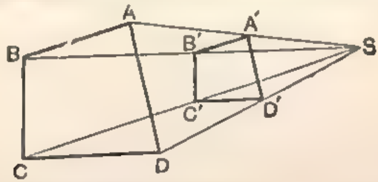


In the latter case it follows also that C′D′ is par[l] to CD, and D′A′ par[l] to DA, and it may be proved as before that AA′, BB′, CC′, DD′ are concurrent ; but here S divides AA′ *internally* in the ratio of corresponding sides, and the position of the figures is *transverse*.

In each case the point S is called a **centre of similarity, or homothetic centre** ; and similar figures so placed are said to be **homothetic**.

Problem 39. [Second Method.]

On a given side to draw a figure similar to a given figure.



Let ABCD be the given figure, and A′B′ the given side ; and let A′B′ correspond to AB.

Construction. Place A′B′ par[l] to AB ; and join AA′, BB′ by lines meeting at S.

Join SC, SD.

Through B′ draw B′C′ par[l] to BC, to meet SC at C′ ;
through C′ draw C′D′ par[l] to CD, to meet SD at D′.

Join A′D′.

Then A′B′C′D′ is the required figure.

The student should prove (i) that A′B′C′D′ is equiangular to ABCD, (ii) that corresponding sides of these figures are proportional. The proof is the converse of Theorem 68.

EXERCISES ON SIMILAR FIGURES.

(Numerical and Graphical.)

1. On a base AB, 6·5 cm. in length, draw a quadrilateral ABCD from the following data:

$$\angle A = 80°, \quad \angle B = 70°, \quad AD = 4{\cdot}4 \text{ cm.}, \quad BC = 3{\cdot}2 \text{ cm.}$$

Taking any convenient point as centre of similarity, make

(i) A reduced copy of ABCD, such that the ratio of each side to the corresponding side of ABCD is 3 : 4.

(ii) An enlarged copy of ABCD, such that the ratio of each side to the corresponding side of ABCD is 5 : 4.

2. Draw a semi-circle on a given diameter AB, and inscribe a square in it, so that two vertices may be on the arc, and the other two on AB. If $AB = 2r$, and the side of the inscribed square $= a$, shew that

$$5a^2 = 4r^2.$$

3. Draw a sector of a circle of radius 2·4″, the central angle being 60°; and inscribe a square in it.

If the radius of the sector $= r$, and the side of the square $= a$, calculate from measurements the ratio $a : r$.

4. In a sector of which the radius = 5 cm., and the central angle = 45°, inscribe a rectangle having its adjacent sides in the ratio 2 : 1.

Prove that two such rectangles can be drawn, and compare by measurement their greater sides.

5. Draw a triangle ABC, making $a = 8$ cm., $b = 7$ cm., and $c = 6$ cm. Working from the vertex A as centre of similarity, inscribe a square in the triangle, so that two of its angular points may be in the base BC, and the other two in AB, AC.

6. Draw a triangle ABC, making $a = 2{\cdot}6''$, B = 110°, C = 35°.

In the triangle ABC inscribe an equilateral triangle, having

(i) one side parallel to BC;

(ii) one side parallel to any given straight line.

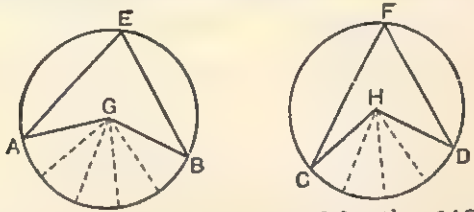
7. In a given triangle ABC inscribe a triangle similar to a given triangle DEF.

In how many ways may this be done?

8. Draw a regular hexagon ABCDEF on a side of 1·2″, and in it inscribe a square having two sides parallel to AB and DE, and its vertices on the remaining sides of the hexagon.

THEOREM 69. [Euclid VI. 33.]

In equal circles, angles, whether at the centres or circumferences, have the same ratio as the arcs on which they stand.



Let ABE, CDF be equal circles; and let the $\angle^s$ AGB, CHD at the centres, and the $\angle^s$ AEB, CFD at the $\bigcirc^{ces}$, stand on the arcs AB, CD.

It is required to prove that

(i) *the* $\angle$AGB : *the* $\angle$CHD = *the arc* AB : *the arc* CD;

(ii) *the* $\angle$AEB : *the* $\angle$CFD = *the arc* AB : *the arc* CD.

Proof. Suppose the arc AB : the arc CD = $m : n$; so that, if the arc AB is divided into m equal parts, then the arc CD may be divided into n such equal parts.

In each circle let radii be drawn to the points of division of the arcs AB, CD.

Then the $\angle^s$ AGB, CHD, in equal circles, are divided into angles which stand on equal arcs, and are therefore all equal.

And of these equal angles the $\angle$AGB contains m,
and the $\angle$CHD contains n;

$\therefore$ the $\angle$AGB : the $\angle$CHD = $m : n$.

Hence the $\angle$AGB : the $\angle$CHD = the arc AB : the arc CD.

And since the $\angle$AEB = one half of the $\angle$AGB; *Theor.* 38.

and the $\angle$CFD = one half of the $\angle$CHD;

$\therefore$ the $\angle$AEB : the $\angle$CFD = the arc AB : the arc CD.

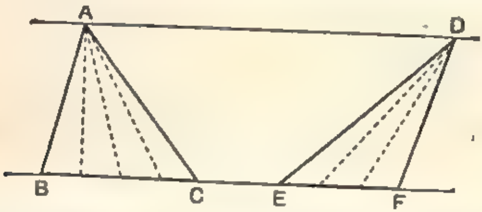
Q.E.D.

COROLLARY. Since *in equal circles, sectors which have equal angles are equal* [Theor. 42, Cor.], it may be proved as above that *the sector* AGB : *the sector* CHD = *the arc* AB : *the arc* CD.

PROPORTION APPLIED TO AREAS.

THEOREM 70. [Euclid VI. 1.]

The areas of triangles of equal altitude are to one another as their bases.



Let ABC, DEF be two triangles of equal altitude, standing on the bases BC, EF.

It is required to prove that

the $\triangle$ABC : *the* $\triangle$DEF = BC : EF.

Proof. Let the triangles be placed so that the bases BC, EF are in the same st. line, and the triangles on the same side of the line.

Join AD; then AD is par[l] to BF. *Def.* 2. p. 99.

Suppose the base BC : the base EF = $m : n$;
so that, if BC is divided into m equal parts, then EF may be divided into n such equal parts.

In each triangle let st. lines be drawn from the vertex to the points of division in the bases BC, EF.

Then the $\triangle^s$ ABC, DEF are divided into triangles which stand on equal bases, and have the same altitude, and are therefore all equal.

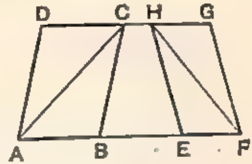
And of these equal $\triangle^s$, the $\triangle$ABC contains m;
and the $\triangle$DEF contains n.

$\therefore$ the $\triangle$ABC : the $\triangle$DEF = $m : n$.

Hence the $\triangle$ABC : the $\triangle$DEF = BC : EF.

Q.E.D.

COROLLARY. *The areas of parallelograms of equal altitude are to one another as their bases.*



For let DB, EG be par^{ms} of the same altitude, standing on the bases AB, EF.

Join AC, HF.

Then

since the par^m DB = twice the △CAB;

and the par^m EG = twice the △HEF;

∴ the par^m DB : the par^m EG = the △CAB : the △HEF

= AB : EF.

ALTERNATIVE PROOF OF THEOREM 70.

Let p represent the altitude of each of the △^{s} ABC, DEF.

Then the area of the △ABC = $\frac{1}{2}$. base × altitude = $\frac{1}{2}$. BC × p;

and the area of the △DEF = ... = $\frac{1}{2}$. EF × p.

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} . BC \times p}{\frac{1}{2} . EF \times p} = \frac{BC}{EF}.$$

EXERCISES.

(*Numerical.*)

1. Two triangles of equal altitude stand on bases of 6·3″ and 5·4″ respectively; if the area of the first triangle is $12\frac{1}{4}$ square inches, find the area of the second.

2. The areas of two triangles of equal altitude have the ratio 24 : 17; if the base of the first is 4·2 cm., find the base of the second to the nearest millimetre.

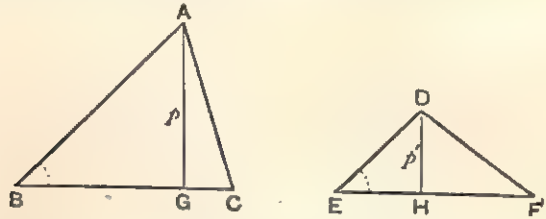
3. Two triangles lying between the same parallels have bases of 16·20 metres and 20·70 metres; find to the nearest square centimetre the area of the second triangle, if that of the first is 50·1204 sq. metres.

4. Two parallelograms whose areas are in the ratio 2·1 : 3·5 lie between the same parallels. If the base of the first is 6·6″ in length, find the base of the second.

5. Two triangular fields lie on opposite sides of a common base; and their altitudes with respect to it are 4·20 chains and 3·71 chains. If the first field contains 18 acres, find the acreage of the whole quadrilateral.

Theorem 71.

If two triangles have one angle of the one equal to one angle of the other, their areas are proportional to the rectangles contained by the sides about the equal angles.



In the $\triangle^s$ ABC, DEF, let the $\angle^s$ at B and E be equal.

It is required to prove that

$$\text{the } \triangle ABC : \text{the } \triangle DEF = AB \,.\, BC : DE \,.\, EF.$$

Let AG and DH be drawn perp. to BC, EF respectively, and denote the lengths of these perp^s by p and p'.

Proof. The $\triangle ABC = \frac{1}{2} BC \,.\, p$; and the $\triangle DEF = \frac{1}{2} EF \,.\, p'$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC \,.\, p}{EF \,.\, p'}. \quad \text{..........................(i)}$$

But since the $\angle B =$ the $\angle E$, and the $\angle G =$ the $\angle H$,

$\therefore$ the $\triangle^s$ ABG, DEH are equiangular to one another; *Theor.* 16.

$$\therefore \frac{p}{p'} = \frac{AB}{DE}. \quad \text{...............(ii)} \quad \textit{Theor. } 62.$$

Substituting for $\frac{p}{p'}$ in (i),

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC \,.\, AB}{EF \,.\, DE};$$

or the $\triangle ABC$: the $\triangle DEF = AB \,.\, BC : DE \,.\, EF$.

Q.E.D.

Corollary. *Similarly it may be shewn that* parallelograms *having one angle of the one equal to one angle of the other are proportional to the rectangles contained by the sides about the equal angles.*

EXERCISES ON AREAS.

(On Theorem 70.)

1. Assuming *the area of a triangle* $= \frac{1}{2}$ *base* $\times$ *altitude*, prove that *triangles on equal bases are proportional to their altitudes.*

Also deduce this result geometrically from Theorem 70.

2. XY is drawn parallel to BC, the base of the triangle ABC, cutting the sides AB, AC in X and Y.

Join BY and CX, and prove, by Theorem 70, that

(i) AX : XB = AY : YC.

(ii) AB : AX = AC : AY.

3. Shew that every quadrilateral is divided by its diagonals into four triangles whose areas are proportionals.

4. If two triangles are on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.

(On Theorem 71.)

5. In two triangles ABC, DEF, the $\angle$B = the $\angle$E. If AB, BC are 2·7″ and 3·5″ respectively, and DE, EF are 2·1″ and 1·8″, shew that

$\triangle$ ABC : $\triangle$ DEF = 5 : 2.

6. The $\triangle^s$ ABC, DEF are equal in area, and the $\angle$B = the $\angle$E. If AB = 5·6 cm., BC = 6·3 cm., DE = 7·2 cm., find EF.

7. In two parallelograms ABCD, EFGH, the $\angle$B = the $\angle$F, and the areas have the ratio 3 : 4. If AB = 4·8 cm., BC = 13·5 cm., EF = 10·8 cm., find FG.

If p and p' denote the perpendiculars drawn from A and E to BC and FG respectively, shew that $p : p' = 4 : 9$.

8. Prove the formula

area of $\triangle = \frac{1}{2}\, ab \sin C$;

and deduce Theorem 71.

9. The $\triangle$ ABC = the $\triangle$ DEF in area; and AB : DE = EF : BC; shew that the $\angle^s$ B and E are equal or supplementary.

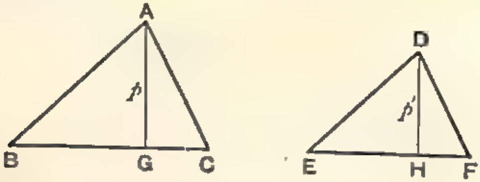
10. The sides AB, AC of the triangle ABC are cut by any straight line at P and Q respectively. By joining PC, and twice applying Theorem 70, shew that

$\triangle$ APQ : $\triangle$ ABC = AP . AQ : AB . AC.

Hence obtain an alternative proof of Theorem 71.

THEOREM 72. [Euclid VI. 19.]

The areas of similar triangles are proportional to the squares on corresponding sides.



Let ABC, DEF be similar triangles, in which BC and EF are corresponding sides.

It is required to prove that

$$\text{the } \triangle \text{ABC} : \text{the } \triangle \text{DEF} = \text{BC}^2 : \text{EF}^2.$$

Let AG and DH be drawn perp. to BC, EF respectively; and denote these perp^s by p and p'.

Proof. The $\triangle \text{ABC} = \frac{1}{2}\text{BC} \cdot p$; and the $\triangle \text{DEF} = \frac{1}{2}\text{EF} \cdot p'$.

$$\therefore \frac{\triangle \text{ABC}}{\triangle \text{DEF}} = \frac{\text{BC} \cdot p}{\text{EF} \cdot p'} \quad \text{.........................(i)}$$

But since the $\angle$ B = the $\angle$ E, from the similar $\triangle$^s ABC, DEF,

and the $\angle$ G = the $\angle$ H, being right angles;

$\therefore$ the $\triangle$^s ABG, DEH are equiangular to one another; *Theor.* 16.

$$\therefore \frac{p}{p'} = \frac{\text{AB}}{\text{DE}} \qquad \textit{Theor. } 62.$$

$$= \frac{\text{BC}}{\text{EF}}, \text{ from the similar } \triangle^{\text{s}} \text{ ABC, DEF.}$$

Substituting for $\frac{p}{p'}$ in (i),

$$\frac{\triangle \text{ABC}}{\triangle \text{DEF}} = \frac{\text{BC} \cdot \text{BC}}{\text{EF} \cdot \text{EF}} = \frac{\text{BC}^2}{\text{EF}^2};$$

or, the $\triangle$ ABC : the $\triangle$ DEF = BC² : EF².

Q.E.D.

EXERCISES ON THE AREAS OF SIMILAR TRIANGLES.

(*Numerical and Graphical.*)

1. In any triangle ABC, the sides AB, AC are cut by a line XY drawn parallel to BC. If AX is one-third of AB, what part is the triangle AXY of the triangle ABC?

2. Two corresponding sides of similar triangles are 3 ft. 6 in. and 2 ft. 4 in. respectively. If the area of the greater triangle is 45 sq. ft., find that of the smaller.

3. The area of the triangle ABC is 25·6 sq. cm., and XY, drawn parallel to BC, cuts AB in the ratio 5 : 3. Find the area of the triangle AXY.

4. Two similar triangles have areas of 392 sq. cm. and 200 sq. cm respectively; find the ratio of any pair of corresponding sides.

5. ABC and XYZ are two similar triangles whose areas are respectively 32 sq. in. and 60·5 sq. in. If XY = 7·7″, find the length of the corresponding side AB.

6. Shew how to draw a straight line XY parallel to BC the base of a triangle ABC, so that the area of the triangle AXY may be nine-sixteenths of that of the triangle ABC.

(*Theoretical.*)

7. ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC, shew that

$$\triangle BAD : \triangle ACD = BA^2 : AC^2.$$

8. A trapezium ABCD has its sides AB, CD parallel, and its diagonals intersect at O. If AB is double of CD, find the ratio of the triangle AOB to the triangle COD.

9. XY is drawn parallel to BC the base of the triangle ABC, if

$$\triangle AXY : \text{fig. } XBCY = 4 : 5,$$

shew that

$$AX : XB = 2 : 1.$$

10. Prove that the areas of similar triangles have the same ratio as the *squares of*

(i) corresponding altitudes;

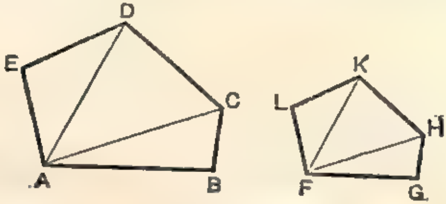
(ii) corresponding medians;

(iii) the radii of their in-circles;

(iv) the radii of their circum-circles.

Theorem 73. [Euclid VI. 20.]

The areas of similar polygons are proportional to the squares on corresponding sides.



Let ABCDE, FGHKL be similar polygons, and let AB, FG be corresponding sides.

It is required to prove that

the polygon ABCDE : *the polygon* FGHKL = $AB^2 : FG^2$.

Join AC, AD, FH, FK.

Proof. Then the $\triangle^s$ ABC, FGH are similar; *Theor.* 67.

also the $\triangle^s$ ACD, FHK are similar;

and the $\triangle^s$ ADE, FKL are similar.

$\therefore$ the $\triangle$ ABC : the $\triangle$ FGH = $AC^2 : FH^2$ *Theor.* 72.

= the $\triangle$ ACD : the $\triangle$ FHK.

Similarly,

the $\triangle$ ACD : the $\triangle$ FHK = $AD^2 : FK^2$

= the $\triangle$ ADE : the $\triangle$ FKL.

Hence $$\frac{\triangle ABC}{\triangle FGH} = \frac{\triangle ACD}{\triangle FHK} = \frac{\triangle ADE}{\triangle FKL}.$$

And in this series of equal ratios, the sum of the antecedents is to the sum of the consequents as each antecedent is to its consequent; *Theor.* V. p. 251.

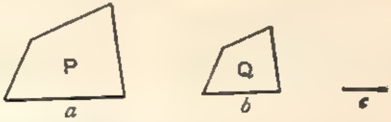
$\therefore$ the fig. ABCDE : the fig. FGHKL = the $\triangle$ ABC : the $\triangle$ FGH

= $AB^2 : FG^2$.

Q.E.D.

COROLLARY 1. Let a, b, c represent three lines in proportion, so that

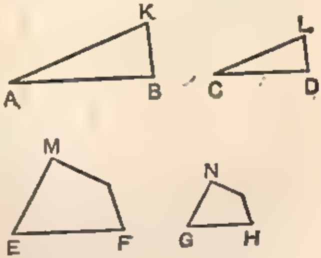
$$\frac{a}{b}=\frac{b}{c}; \text{ and consequently } b^2=ac.$$



Now suppose similar figures P and Q to be drawn on a and b as corresponding sides,

then $$\frac{\text{Fig. P}}{\text{Fig. Q}}=\frac{a^2}{b^2}=\frac{a^2}{ac}=\frac{a}{c}.$$

Hence if three straight lines are proportionals, and any similar figures are drawn on the first and second as corresponding sides, then the fig. on the first : *the fig. on the* second = *the* first : *the* third.



COROLLARY 2. Let

AB : CD = EF : GH ;

and let similar figures KAB, LCD be similarly described on AB, CD, and also let similar figures MF, NH be similarly described on EF, GH.

Then since $$\frac{AB}{CD}=\frac{EF}{GH}; \quad \therefore \frac{AB^2}{CD^2}=\frac{EF^2}{GH^2}.$$

But the fig. KAB : the fig. LCD = $AB^2 : CD^2$; *Theor.* 73.

and the fig. MF : the fig. NH = $EF^2 : GH^2$.

$\therefore$ the fig. KAB : the fig. LCD = the fig. MF : the fig. NH.

Hence if four straight lines are proportional, and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair on the third and fourth, these figures are proportional.

EXERCISES ON THE AREAS OF SIMILAR FIGURES.

(Numerical and Graphical.)

1. Shew how to draw a straight line XY parallel to the base BC of a triangle ABC, so that the area of the triangle AXY may be four-ninths of the triangle ABC.

2. The sides of a triangle are 2·0″, 2·5″, 3·2″; find the sides of a similar triangle of three times the area.

[The results are to be given to the nearest hundredth of an inch.]

3. Two similar triangles have areas in the ratio 13·69 : 16·81, and an altitude of the greater is 10 ft. 3 in. Find the corresponding altitude of the other.

4. ABC is a triangle whose area is 16 sq. cm.; and XY is drawn parallel to BC, dividing AB in the ratio 3 : 5; if BY is joined, find the area of the triangle BXY.

5. One-fifth of the area of the triangle ABC is cut off by a line XY drawn parallel to BC. If BC = 10 cm., find XY to the nearest millimetre.

6. The area of a regular pentagon on a side of 2·5″ is approximately $10\frac{3}{4}$ sq. in.; find the area of a similar figure on a side of 3·0″.

7. The length of a rectangular area is 10·8 metres, and the ratio of the length to the breadth is 12 : 5. Find the length and breadth of a similar rectangle containing one-ninth of the area.

8. In the plan of a certain field, 1″ represents 66 yards; if the area of the plan is found to be 100 sq. in., find the area of the field in acres.

Explain why in this example the *shape* of the field is immaterial.

9. An estate is represented on a plan by a quadrilateral ABCD drawn to the scale of 25″ to the mile. If AC = 20″ and the offsets from AC to B and D measure 24″ and 26″ respectively, find the acreage of the estate.

10. A field of 1·89 hectares is represented on a plan by a triangle whose sides measure 13 cm., 14 cm., and 15 cm. On what scale is the plan drawn?

EXERCISES ON THE AREAS OF SIMILAR FIGURES.

(Theoretical.)

1. If ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC, shew that

(i) $BC^2 : BA^2 = BC : BD$; [Theor. 73, Cor. 1.]

(ii) $BC^2 : CA^2 = BC : CD$.

Hence deduce $BC^2 = BA^2 + AC^2$.

2. A triangle ABC is bisected by a straight line XY drawn parallel to the base BC. Determine the ratio AX : AB.

Hence shew how to bisect a triangle by a straight line drawn parallel to the base.

3. Two circles have external contact at A, and a common tangent, touching them at B and C, meets the line of centres at S. If AB, AC are joined, shew that

$$\triangle SBA : \triangle SAC = SB : SC.$$

4. Two circles intersect at A and B, and at A tangents are drawn, one to each circle, meeting the circumferences at C and D. If AB, CB and BD are joined, shew that

$$\triangle CBA : \triangle ABD = CB : BD.$$

5. DEF is the pedal triangle [see p. 207] of the triangle ABC; prove that $\triangle ABC : \triangle DBF = AB^2 : BD^2$.

Hence shew that

$$\text{fig. } AFDC : \triangle DBF = AD^2 : BD^2.$$

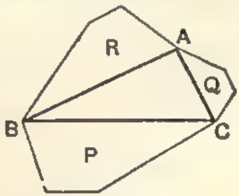
6. In a given triangle ABC a second triangle is inscribed by joining the middle points of the sides. In this inscribed triangle a third is inscribed in like manner, and so on. What fraction is the fourth triangle of the triangle ABC?

7. A regular hexagon is drawn on a side of a centimetres, and a second hexagon is inscribed in it by joining the middle points of the sides in order. In like manner a third hexagon is inscribed in the second, and so on. Find the ratio of the first hexagon to the fifth.

8. *Shew that the areas of two similar cyclic figures are proportional to the squares of he diameters of their circum-circles.* [Euclid XII. 1.]

THEOREM 74. [Euclid VI. 31.]

In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the two similar and similarly described figures on the sides containing the right angle.



Let ABC be a right-angled triangle of which BC is the hypotenuse; and let P, Q, R be similar and similarly described figures on BC, CA, AB respectively.

It is required to prove that

the fig. R + the fig. Q = the fig. P.

Proof. Since AB and BC are corresponding sides of the similar figs. R and P,

$$\therefore \frac{\text{fig. R}}{\text{fig. P}} = \frac{AB^2}{BC^2} \ldots\ldots\ldots\ldots(\text{i}) \quad \textit{Theor. 73.}$$

In like manner,

$$\frac{\text{fig. Q}}{\text{fig. P}} = \frac{AC^2}{BC^2} \ldots\ldots\ldots\ldots(\text{ii})$$

Adding the equal ratios on each side in (i) and (ii)

$$\frac{\text{fig. R} + \text{fig. Q}}{\text{fig. P}} = \frac{AB^2 + AC^2}{BC^2}.$$

But

$$AB^2 + AC^2 = BC^2; \qquad \textit{Theor. 29.}$$

$\therefore$ the fig. R + the fig. Q = the fig. P. Q.E.D.

COROLLARY. *The area of a circle drawn on the hypotenuse of a right-angled triangle as diameter is equal to the sum of the circles similarly drawn on the other sides.*

For the areas of circles are proportional to the squares on their diameters. [Page 203.]

EXERCISES.

(*Miscellaneous.*)

1. In a triangle ABC, right-angled at A, AD is drawn perpendicular to the hypotenuse. Shew that

(i) $BA^2 = BC \cdot BD$; (ii) $CA^2 = CB \cdot CD$.

Hence deduce Theorem 29, namely,

$$BC^2 = BA^2 + AC^2.$$

2. In the diagram of Theorem 74, draw AD perpendicular to BC; hence prove that, if the fig. P = the △ ABC, then

(i) the fig. Q = the △ ADC; (ii) the fig. R = the △ ADB.

3. In the diagram of Theorem 74, if AB : AC = 8 : 5, and if the fig. P = 8·9 sq. cm., find the areas of the figs. Q and R.

4. BY and CZ are medians of the triangle ABC, and YZ is joined. Find the ratio of the triangle BOC to the triangle YOZ. [See p. 97.]

5. ABC is an isosceles triangle, the equal sides AB, AC each measuring 3·6″. From a point D in AB, a straight line DE is drawn cutting AC produced at E, and making the triangle ADE equal in area to the triangle ABC. If AD = 1·8″, find AE.

6. AB is a diameter of a circle, and two chords AP, AQ are produced to meet the tangent at B in X and Y.

Shew that (i) the △s APQ, AYX are similar;
(ii) the four points P, Q, Y, X are concyclic.

7. In the triangle ABC, the angle A is externally bisected by a line which meets the base produced at D and the circum-circle at E: shew that

$$AB \cdot AC = AE \cdot AD.$$

8. When is a straight line said to be divided *in extreme and mean ratio?*

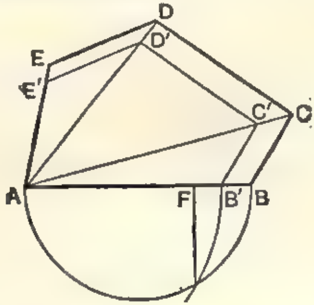
If a line 10 cm. in length is so divided, find the approximate lengths of the segments, and check your work graphically.

9. Draw an isosceles triangle equal in area to a triangle ABC, and having its vertical angle equal to the angle A.

10. On a given base draw an isosceles triangle equal in area to a given triangle ABC.

Problem 40.

To draw a figure similar to a given rectilineal figure, and equal to a given fraction of it in area.



Let ABCDE be the given figure, to which a similar figure is to be drawn, having its area a given fraction (say *three-fourths*) of that of the fig. ABCDE.

Construction. Make AF *three-fourths* of AB. *Prob.* 7.

From AB cut off AB′ the mean proportional between AF and AB. *Prob.* 38 *Note.*

On AB′ draw the fig. AB′C′D′E′ similar to the fig. ABCDE. *Prob.* 39.

Then the fig. AB′C′D′E′ $=\frac{3}{4}$ of the fig. ABCDE.

Proof. By construction, $AB'^2 = AF \cdot AB$.

Now the figs. ABCDE, AB′C′D′E′ are similar, and AB, AB′ are corresponding sides;

$$\therefore \frac{\text{fig. } AB'C'D'E'}{\text{fig. } ABCDE} = \frac{AB'^2}{AB^2} \qquad \textit{Theor. } 71.$$

$$= \frac{AF \cdot AB}{AB^2}$$

$$= \frac{AF}{AB} = \frac{3}{4}.$$

EXERCISES.

1. Divide a triangle ABC into two parts of equal area by a line XY drawn parallel to the base BC and cutting the other sides at X and Y.

Find (i) by calculation, (ii) by measurement, the ratio AX : AB.

2. Divide a triangle ABC into three parts of equal area by lines PQ, XY drawn parallel to the base BC. If P and X lie on AB, prove that

$$\frac{AP}{1}=\frac{AX}{\sqrt{2}}=\frac{AB}{\sqrt{3}}.$$

Hence shew how a triangle may be divided into n equal parts by lines drawn parallel to one side.

3. Draw a rectangle of length 8 cm., and breadth 5 cm. Then draw a similar rectangle of one-third the area.

Measure its length to the nearest millimetre, and verify your result by calculation.

4. Draw a quadrilateral ABCD from the following data;

the $\angle A=90°$; AB=BC=8 cm.; AD=DC=6 cm.

Draw a similar quadrilateral to contain an area of 36 sq. cm., and find to the nearest millimetre the length of the side corresponding to AB.

5. Divide a circle of radius 3″ into three equal parts by means of two concentric circles.

6. *Draw a rectilineal figure equal in area to a given figure* E, *and similar to a given figure* S. [Euclid VI. 25.]

[First replace the given figures E and S by equivalent squares (see Problems 19 and 32). Let the sides of these squares be a and b respectively, and let s be one of the sides of S.

Find p, a fourth proportional, to b, a, s, so that $b:a=s:p$.

On p draw a figure P similar to the figure S, so that p and s are corresponding sides. Then P is the figure required;

for $$\frac{P}{S}=\frac{p^2}{s^2}=\frac{a^2}{b^2}=\frac{E}{S}.$$

$\therefore$ the fig. P = the fig. E.]

RECTANGLES IN CONNECTION WITH CIRCLES.

NOTE. We here give a simple proof of Theorems 57 and 58 brought under a single enunciation. [See NOTE p. 234.]

THEOREM 75. [Euclid III. 35 and 36.]

If any two chords of a circle cut one another internally or externally, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

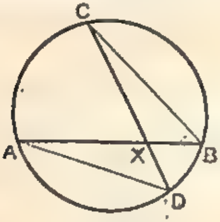


Fig. 1.

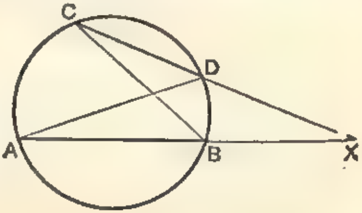


Fig. 2.

In the ⊙ ABC, let the chords AB, CD cut one another at X, internally in Fig. 1, and externally in Fig. 2.

It is required to prove in both cases that

the rect. XA, XB = *the rect.* XC, XD.

Join AD, BC.

Proof. In the $\triangle^s$ AXD, CXB,

the ∠AXD = the ∠CXB, being opp. vert. $\angle^s$ in Fig. 1, and the same angle in Fig. 2;

and the ∠A = the ∠C, being $\angle^s$ at the $\bigcirc^{ce}$, standing on the same arc BD;

∴ the remaining angles are equal; *Theor.* 16.

hence the $\triangle^s$ AXD, CXB are equiangular,

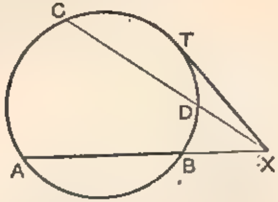
$$\therefore \frac{XA}{XC} = \frac{XD}{XB};$$

$$\therefore XA \cdot XB = XC \cdot XD;$$

that is, the rect. XA, XB = the rect. XC, XD.

Q.E.D.

COROLLARY. *If from an external point a secant and a tangent are drawn to a circle, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.*



Let XBA be a secant, and XT a tangent drawn to the ⊙ABC from the point X.

It is required to prove that $XA \cdot XB = XT^2$.

Let XDC be a second secant;

then $XA \cdot XB = XC \cdot XD$, *Theor.* 75. Fig. 2.

and this is true for all positions of the line XDC.

Now let XDC turn about X away from the centre, so that the points C and D continually approach one another and ultimately coincide at T;

then XDC becomes the tangent XT,

and $XC \cdot XD$ becomes $XT \cdot XT$, or XT^2,

$\therefore$, ultimately, $XA \cdot XB = XT^2$.

EXERCISES FOR SQUARED PAPER.

1. From the point (1·7, 0) as centre, a circle is drawn to touch OY at O, and cutting OX at A. If *any* line is drawn from A to cut OY at Q and the circle at P, shew that $AP \cdot AQ$ is constant, and find its value when 1″ is taken as the unit of length.

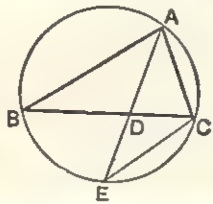
2. A circle of radius 10 is drawn from centre C (5, 6). If TT′ is the chord of contact of tangents from P (29, 16), and if PC meets TT′ in Q find the value of

(i) $CQ \cdot CP$; (ii) $PQ \cdot CP$; and (iii) the length of TT′.

3. From centres (·3, 0) (2, 0) circles of radii 2·6 and 2·5 respectively are drawn. Find the coordinates of their common points, and the length of their common chord. Also find the length of a tangent to each circle from the point (1·3, 3·4). Verify your results by measurement.

* THEOREM 76.

If the vertical angle of a triangle is bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.



Let ABC be a triangle, having the ∠BAC bisected by AD.

It is required to prove that

the rect. AB, AC = *the rect.* BD, DC + *the sq. on* AD.

Suppose a circle circumscribed about the △ABC; and let AD be produced to meet the ○^{ce} at E.

Join EC.

Proof. Then in the △^{s} BAD, EAC,

because the ∠BAD = the ∠EAC,

and the ∠ABD = the ∠AEC in the same segment;

∴ the remaining ∠BDA = the remaining ∠ECA;

that is, the △^{s} BAD, EAC are equiangular to one another;

$$\therefore \frac{AB}{AE} = \frac{AD}{AC}. \qquad \textit{Theor. } 62.$$

Hence $AB \,.\, AC = AE \,.\, AD$

$= (AD + DE)AD$

$= AD^2 + AD \,.\, DE.$

But $AD \,.\, DE = BD \,.\, DC$; *Theor.* 75.

∴ the rect. AB, AC = the rect. BD, DC + the sq. on AD.

Q.E.D.

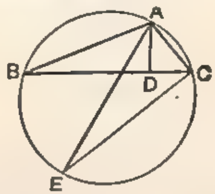
EXERCISE.

If the vertical angle BAC is bisected *externally* by AD, shew that

$AB \,.\, AC = BD \,.\, DC - AD^2$.

* Theorem 77.

If from the vertical angle of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle.



In the $\triangle$ ABC, let AD be the perp. from A to the base BC; and let AE be a diameter of the circum-circle.

It is required to prove that

the rect. AB, AC = *the rect.* AE, AD.

Join EC.

Proof. Then in the $\triangle^s$ BAD, EAC,

the rt. angle BDA = the rt. angle, ECA, in the semicircle ECA,

and the $\angle$ ABD = the $\angle$ AEC, in the same segment;

$\therefore$ the remaining $\angle$ BAD = the remaining $\angle$ EAC;

that is, the $\triangle^s$ BAD, EAC are equiangular to one another.

$$\therefore \frac{AB}{AE} = \frac{AD}{AC}; \qquad \textit{Theor. } 62.$$

Hence $\qquad$ AB . AC = AE . AD;

or $\qquad$ the rect. AB, AC = the rect. AE, AD.

Q.E.D.

Note. Let a, b, c denote the sides of the $\triangle$ ABC, R its circum-radius, and p the perp. AD.

Then since $\qquad$ AE . AD = AB . AC,

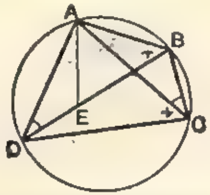
$$2R \cdot p = cb;$$

$$\therefore R = \frac{bc}{2p}$$

$$= \frac{abc}{2ap} = \frac{abc}{4\Delta}.$$

THEOREM 78. [Ptolemy's Theorem.]

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quadrilateral inscribed in a circle, and let AC, BD be its diagonals.

It is required to prove that

the rect. AC, BD = *the rect.* AB, CD + *the rect.* BC, DA.

Make the $\angle$ DAE equal to the $\angle$ BAC;
to each add the $\angle$EAC,
then the $\angle$ DAC = the $\angle$ EAB.

Proof. Then in the $\triangle$s EAB, DAC,
the $\angle$ EAB = the $\angle$ DAC,
and the $\angle$ ABE = the $\angle$ ACD in the same segment;
$\therefore$ the $\triangle$s EAB, DAC are equiangular to one another; *Theor.* 16.

$$\therefore \frac{BA}{CA} = \frac{BE}{CD}; \qquad \textit{Theor. } 62.$$

hence AB . CD = AC . BE. ....(i)

Again in the $\triangle$s DAE, CAB,
the $\angle$ DAE = the $\angle$ CAB,
and the $\angle$ ADE = the $\angle$ ACB, in the same segment;
$\therefore$ the $\triangle$s DAE, CAB are equiangular to one another;

$$\therefore \frac{DA}{CA} = \frac{DE}{CB};$$

hence BC . DA = AC . DE. ....(ii)

Adding the equal rectangles on each side in (i) and (ii)

AB . CD + BC . DA = AC . BE + AC . DE
= AC (BE + DE)
= AC . BD.

Q.E.D.

EXERCISES.

1. ABC is an isosceles triangle, and on the base, or base produced, any point X is taken: shew that the circumscribed circles of the triangles ABX, ACX are equal.

2. From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn perpendicular to AB, AC respectively, and intersecting at D: shew that

$$BC \,.\, AD = 2AB \,.\, DB.$$

3. If the diagonals of a quadrilateral inscribed in a circle are at right angles, the sum of the rectangles contained by the opposite sides is double the area of the figure.

4. ABCD is a quadrilateral inscribed in a circle, and the diagonal BD bisects AC: shew that

$$AD \,.\, AB = DC \,.\, CB.$$

5. If the vertex A of a triangle ABC is joined to any point in the base, it will divide the triangle into two triangles such that their circumscribed circles have radii in the ratio of AB to AC.

6. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

7. Two triangles of equal area are inscribed in the same circle: shew that the rectangle contained by any two sides of the one is to the rectangle contained by any two sides of the other as the base of the second is to the base of the first.

8. P is a point on the arc BC of the circum-circle of an equilateral triangle ABC. If P is joined to A, B, and C, show that

$$PB + PC = PA.$$

9. ABCD is a quadrilateral inscribed in a circle, and BD bisects the angle ABC: if the points A and C are fixed on the circumference of the circle, and B is variable in position, shew that

$AB + BC : BD$ is a constant ratio.

10. From the formula $R = \dfrac{abc}{4\Delta}$ (see NOTE, p. 303) find the value of R when the sides of the triangle are as follows:

(i) 21″, 20″, 13″; (ii) 30 ft., 25 ft., 11 ft.

Draw the triangles to a convenient scale and check your work by measurement.

MISCELLANEOUS THEORETICAL EXAMPLES ON PARTS I.–V.

1. Two circles whose centres are C and D intersect at A and B; and a straight line PAQ is drawn through A and terminated by the circumferences: prove that

(i) the $\angle$PBQ = the $\angle$CAD;
(ii) the $\angle$BPC = the $\angle$BQD.

2. AB is a given diameter of a circle, and CD is any chord parallel to AB; if any point X in AB is joined to the extremities of CD, shew that

$$XC^2 + XD^2 = XA^2 + XB^2.$$

3. The opposite sides of a cyclic quadrilateral are produced to meet: shew that the bisectors of the two angles so formed are perpendicular to one another.

4. Given the vertical angle, one of the sides containing it, and the length of the perpendicular from the vertex on the base: construct the triangle.

5. A, B, C are three points in order in a straight line: find a point P in the straight line so that PB may be a mean proportional between PA and PC.

6. Through D, any point in the base of a triangle ABC, straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: shew that the triangle AEF is a mean proportional between the triangles FBD, EDC.

---

7. PQ is a fixed chord in a circle, and PX, QY any two parallel chords through P and Q; shew that XY touches a fixed concentric circle.

8. Two circles touch each other at C, and straight lines are drawn through C at right angles to one another, meeting the circles at P, P′ and Q, Q′ respectively: if the straight line which joins the centres is terminated by the circumferences at A and A′ shew that

$$P'P^2 + Q'Q^2 = A'A^2.$$

9. AE bisects the vertical angle of the triangle ABC and meets the base in E. If d, d' are the diameters of the circum-circles of the triangles ABE, ACE, shew that

$$d : d' = BE : EC.$$

10. AB, AC are chords of a circle; a line parallel to the tangent at A cuts AB, AC in D and E respectively: shew that

$$AB . AD = AC . AE.$$

11. If a straight line is divided at two given points, determine a third point such that its distances from the extremities may be proportional to its distances from the given points.

12. Given the feet of the perpendiculars drawn from the vertices on the opposite sides: construct the triangle.

---

13. If a quadrilateral can have one circle inscribed in it, and another circumscribed about it, shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to one another.

14. Two equal circles move between two straight lines placed at right angles, so that each line is touched by one circle, and the two circles touch one another: find the locus of the point of contact.

15. AB is a diameter of a given circle; and AC, BD, two chords on the same side of AB intersect at E: shew that the circle which passes through D, E, C cuts the given circle orthogonally. [See Def. p. 330.]

16. If four circles are described to touch every three sides of a quadrilateral, shew that their centres are concyclic.

17. AB is a straight line divided at C and D so that

$$AB : AC = AC : AD;$$

from A a line AE is drawn in any direction and equal to AC; shew that BC and CD subtend equal angles at E.

18. Given the vertical angle, the ratio of the sides containing it, and the diameter of the circumscribing circle, construct the triangle.

---

19. O is a fixed point, and OP is any line drawn to meet a fixed straight line in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

20. O is a fixed point, and OP is any line drawn to meet the circumference of a fixed circle in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

21. Two equal circles intersect at A and B; and from C, any point on the circumference of one of them, a perpendicular is drawn to AB, meeting the other circle at O and O′; shew that either O or O′ is the orthocentre of the triangle ABC. Distinguish between the two cases.

22. Three equal circles pass through the same point A, and their other points of intersection are B, C, D: shew that of the four points A, B, C, D, each is the orthocentre of the triangle formed by joining the other three.

23. From a given point without a circle draw a straight line to the concave circumference so as to be bisected by the convex circumference. When is this problem impossible?

24. Given the base, the altitude, and the radius of the circum-circle: construct the triangle.

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25. Given the base of a triangle and the sum of the remaining sides: find the locus of the foot of the perpendicular from one extremity of the base on the bisector of the exterior vertical angle.

26. Construct a triangle having given either the three ex-centres, or the in-centre and two ex-centres.

27. If O is the orthocentre of a triangle ABC, shew that

$$AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2 = d^2,$$

where d is the diameter of the circum-circle.

28. If C is the middle point of an arc of a circle whose chord is AB, and D is any point in the conjugate arc; shew that

$$AD + DB : DC = AB : AC.$$

29. D is a point in the side AC of the triangle ABC, and E is a point in AB. If BD, CE divide each other into parts in the ratio 4 : 1, then D, E divide CA, BA in the ratio 3 : 1.

30. If the perpendiculars from two fixed points on a straight line passing between them are in a given ratio, the straight line must pass through a third fixed point.

---

31. From the vertex A of any triangle ABC draw a line meeting BC produced in D so that AD may be a mean proportional between the segments of the base.

32. Two circles touch internally at O; AB a chord of the larger circle touches the smaller in C which is cut by the lines OA, OB in the points P, Q: shew that OP : OQ = AC : CB.

33. AB is any chord of a circle; AC, BC are drawn to any point C in the circumference and meet the diameter perpendicular to AB at D, E: if O is the centre, shew that the rect. OD, OE is equal to the square on the radius.

34. YD is a tangent to a circle drawn from a point Y in the diameter AB produced; from D a perpendicular DX is drawn to the diameter; shew that the points X, Y divide AB internally and externally in the same ratio.

35. Determine a point in the circumference of a circle, from which lines drawn to two other given points shall have a given ratio.

36. Given the base, and the position of the bisector of the vertical angle: construct the triangle.

---

37. EA, EA′ are diameters of two circles touching each other externally at E; a chord AB of the former circle, when produced, touches the latter at C′, while a chord A′B′ of the latter touches the former at C: prove that

$$AB \cdot A'B' = 4BC' \cdot B'C.$$

38. From a given external point draw a straight line to cut off a quadrant from a given circle.

39. Shew that the straight lines joining the circum-centre of a triangle to its vertices are perpendicular to the corresponding sides of the pedal triangle.

40. P is any point on the circum-circle of a triangle ABC; and perpendiculars PD, PE are drawn to the sides BC, CA. Find the locus of the circum-centre of the triangle PDE.

41. P is any point on the circum-circle of a triangle ABC: shew that the angle between Simson's Line for the point P and the side BC is equal to the angle between AP and that diameter of the circum-circle which passes through A.

42. Given the base, the vertical angle, and the difference of the angles at the base: construct the triangle.

---

43. Shew that the circles circumscribed about the four triangles formed by two pairs of intersecting straight lines meet in a point.

44. Shew that the orthocentres of the four triangles formed by two pairs of intersecting straight lines are collinear.

45. Of all polygons of a given number of sides, which can be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.

46. On a straight line PAB, two points A and B are marked and the line PAB is made to revolve round the fixed extremity P. C is a fixed point in the plane in which PAB revolves; prove that if CA and CB are joined, and the parallelogram CADB is completed, the locus of D will be a circle.

47. Describe an equilateral triangle equal to a given isosceles triangle.

48. Given the vertical angle of a triangle in position and magnitude, and the sum of the sides containing it: find the locus of the circum-centre.

---

49. ABC is any triangle, and on its sides equilateral triangles are described externally: if X, Y, Z are the centres of their in-circles shew that the triangle XYZ is equilateral.

50. In a given circle inscribe a triangle so that two sides may pass through two given points and the third side be parallel to a given straight line.

51. In a given circle inscribe a triangle so that the sides may pass through three given points.

52. A, B, X, Y are four points in a straight line, and O is such a point in it that the rectangle OA, OY is equal to the rectangle OB, OX; if a circle is described with centre O and radius equal to a mean proportional between OA and OY, shew that at every point on this circle AB and XY will subtend equal angles.

53. Find the locus of a point which moves so that its distances from two intersecting straight lines are in a given ratio.

54. If S, I, I_1, are the circum-centre, in-centre, and an ex-centre of a triangle, and R, r, r_1 the radii of the corresponding circles, and if N is the centre of the nine-points circle, prove that

(i) $SI^2 = R^2 - 2Rr$; (ii) $SI_1^2 = R^2 + 2Rr_1$;

(iii) $NI = \frac{1}{2}R - r$; (iv) $NI_1 = \frac{1}{2}R + r_1$.

MISCELLANEOUS THEOREMS AND EXAMPLES.

I. SOME CONSTRUCTIONS OF CIRCLES.

EXAMPLE 1. *Draw a circle to touch a given circle* (C), *and also to touch a given straight line* PQ *at a given point* A.

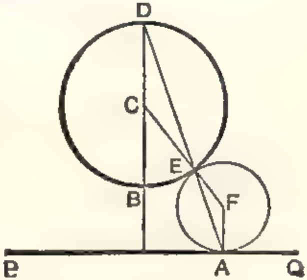
Construction. At A draw AF perp. to PQ:

then the centre of the required ⊙ must lie in AF.

Take C the centre of the given ⊙, and draw the diam. BD perp. to PQ.

Join A to one extremity D, of the diameter, cutting the Oce at E.

Join CE, and produce it to cut AF at F.



Then F *is the centre, and* FA *the radius of the required circle.*

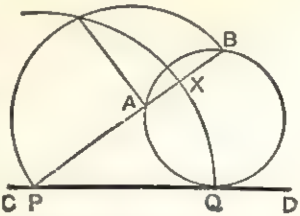
[Supply the proof; and shew that a second solution is obtained by joining AB, and producing it to meet the Oce.]

EXAMPLE 2. *Draw a circle to pass through two given points* A *and* B, *and to touch a given straight line* CD.

Construction. Join BA, and produce it to meet CD at P.

Find PX the mean proportional between PA and PB. *Prob. 38, Note.*

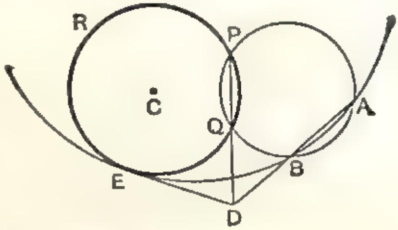
From PD (or PC) cut off PQ equal to PX.



Then the circle drawn through A, B, *and* Q [Prob. 25] *will touch* CD *at* Q.

[Supply the proof; and shew that there are in general two solutions. Modify the construction to meet the case when AB is parallel to CD.]

EXAMPLE 3. *Draw a circle to pass through two given points* A *and* B, *and to touch a given circle* (C).



Construction. Through A and B draw *any* circle to cut the given circle at P and Q.

Join AB, PQ and produce them to meet at D.

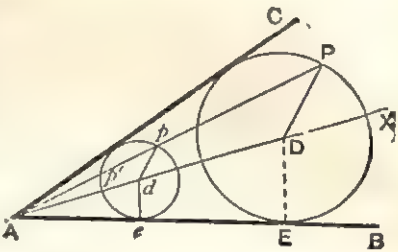
From D draw a tangent DE to the given circle.

Then the circle drawn through A, B, E *will touch the given circle at* E.

[Supply the proof from Theorems 58 and 59; and shew that there are in general two solutions.

Modify the construction to meet the case when the straight line bisecting AB at right angles passes through C.]

EXAMPLE 4. *Draw a circle to pass through a given point* P, *and to touch two given straight lines* AB, AC.



Construction. Draw AX bisecting the ∠BAC. Then all circles touching AB, AC have their centres in AX.

From any point *d* in AX draw *de* perp. to AB; hence with *d* as centre, draw a circle touching AB and AC.

Join AP, cutting the ⊙ (*d*) at *p*, *p′*.

Join *pd*; and through P draw PD par^{l} to *pd*, cutting AX at D.

Then D *is the centre, and* DP *the radius of the required circle touching* AB *and* AC.

[Draw DE perp. to AB. The proof is obtained by shewing that DE = DP, by means of the similar △^{s} ADE, A*de*, and the similar △^{s} ADP, A*dp*.

Shew that a second solution may be obtained by joining *dp′*, and proceeding as before.

Modify the construction to meet the case when the given lines are parallel.]

EXERCISES FOR SQUARED PAPER.

1. Given a circle of radius 10 having its centre at the origin, draw a circle to touch the given circle and also touch the x-axis at the point (20, 0).

Shew that two *equal* circles can be so drawn. Calculate the radius of that in the first quadrant and the coordinates of its point of contact with the given circle.

2. Given a circle of radius 10 having its centre at the origin, draw a circle to touch the given circle at the point (6, 8), and also to touch the y-axis.

Shew that two such circles can be drawn. Find their radii and points of contact with the y-axis.

3. Draw a quadrant of a circle of radius 2″, and inscribe a circle in it. Shew that the radius of the inscribed circle is the positive root of the equation $r^2+4r-4=0$.

Obtain the radius by calculation and by measurement.

4. Shew that two circles can be drawn to touch both axes of coordinates and to pass through the point (2″, 2″); and prove that their radii are given by the quadratic $r^2+4r\sqrt{2}-8=0$.

Draw the smaller of these circles and obtain its radius by measurement.

5. Join the points (2″, 0) and (0, 3″); also join the points (3″, 0) and (0, 2″); then draw a circle to touch the joining lines and to pass through the origin.

6. Within an equilateral triangle on a side of 3·0″ draw three equal circles each to touch two sides of the triangle and the other two circles.

If r is the radius of one of these circles, shew that

$$r(\tan 60^\circ+1)=\frac{3}{2}.$$

Hence find r to the nearest hundredth of an inch.

7. Within a circle of radius 2·0″ draw three equal circles each to touch the other two and the given circle.

If r is the radius of one of these equal circles shew that

$$r(1+\operatorname{cosec} 60^\circ)=2.$$

Hence find r to the nearest hundredth of an inch.

II. MAXIMA AND MINIMA.

When a line, angle, or figure varying under specified conditions, gradually changes its position and magnitude, we may be required to note if any situations exist in which, after increasing, it begins to decrease: or, after decreasing, to increase. In such situations the magnitude is said to have reached a maximum or minimum value. We propose here to deal with problems in which the variable magnitude admits of only *one* transition from an increasing to a decreasing state—and vice versâ: so that for our present purpose the maximum is actually the greatest, and the minimum actually the least value that the variable magnitude can take.

Two hints towards the solution of such problems may be given.

(i) Since a variable geometrical magnitude reaches its maximum or minimum value at a *turning point*, towards which the magnitude may mount or descend from either side, it is natural to expect a maximum or minimum value when the magnitude assumes a *symmetrical* form or position; and this is usually found to be the case.

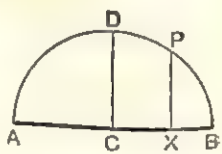
Example I. *Divide a straight line* AB *internally so that the rectangle contained by the two segments may be a maximum.*

Bisect AB at C, and on AB draw a semi-circle.

Take *any* point X in AB; and draw XP perp. to AB to cut the Oce at P.

Then $AX \cdot XB = PX^2$. *Prob.* 32.

Now PX is greatest when it coincides with the radius CD;



$\therefore$ AX . XB is a maximum, when X is the mid-point of AB.

Observe that in this case the maximum is reached when PX occupies the *symmetrical position* in which it bisects AB at right angles.

(ii) Again we can find when a geometrical magnitude assumes its maximum or minimum value, if we can discover a construction for drawing the magnitude so that it may have an *assigned* value: for we may then examine between what limits the assigned value must lie in order that the construction may be possible; and the higher or lower limit will give the maximum or minimum sought for.

It has been pointed out that if under certain conditions existing among the data, *two* solutions of a problem are possible, and under other conditions, *no* solution exists, there will always be some intermediate condition under which the two solutions combine in a *single* solution. [See page 94.]

In these circumstances this single solution will be found to correspond to the maximum or minimum value of the magnitude to be constructed.

EXAMPLE 2. *To find at what point in* CD, *a given straight line of indefinite length, the angle subtended by a finite line* AB *is a maximum.*

First find at what point in CD a *given* angle is subtended by AB. This is done as follows:

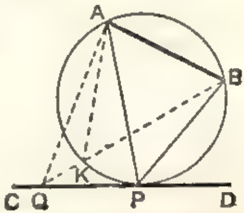
On AB draw a segment of a circle containing an angle equal to the given angle. *Problem* 24.

If the arc of this segment intersects CD, *two* points in CD are found at which AB subtends the given angle: but if the arc does not meet CD, *no* solution is given.

In accordance with the principles explained above, we expect that the maximum angle is determined when the arc *touches* CD, that is, meets it at two coincident points. This we shall prove to be the case.

Draw a circle through A and B to touch CD; and let P be the point of contact. Ex. 2. p. 311.

Then the ∠APB *is greater than any other angle subtended by* AB *at a point in* CD *on the same side of* AB *as* P.



Proof. For take *any* other point Q in CD on the same side of AB as P; and join AQ, QB.

Let BQ meet the circle at K. Join AK.

Then the ∠AKB = the ∠APB, in the same segment.

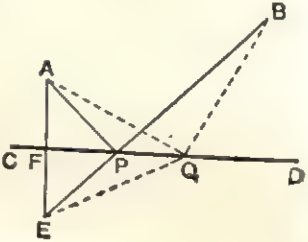
But the ext. ∠AKB is greater than the int. opp. ∠AQB;

∴ the ∠APB is greater than the ∠AQB.

Hence the ∠APB is a maximum.

NOTE. Two circles may be described to pass through A and B, and to touch CD, the points of contact being on opposite sides of AB; hence two points in CD may be found such that the angle subtended by AB at each of them is greater than the angle subtended at any other point in CD *on the same side of* AB.

EXAMPLE 3. *In* CD, *a straight line of indefinite length, find a point* P *such that the sum of its distances from two points* A, B *on the same side of* CD *is a minimum.*



Draw AF perp. to CD; and produce AF to E, making FE equal to AF.

Join EB, cutting CD at P. Join AP.

Then AP + PB *is a minimum.*

Proof. Take *any* other point Q in CD, and join AQ, BQ, EQ.

Now the $\triangle^s$ AFP, EFP are congruent,

$\therefore$ AP = EP.

Similarly AQ = EQ.

And in the $\triangle$ EQB, EQ + QB is greater than EB;

hence AQ + QB is greater than EB,

that is, greater than AP + PB.

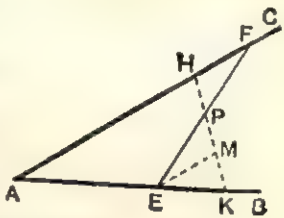
Thus AP + PB is a minimum.

NOTE. It follows that the $\angle$APF = the $\angle$EPF *Theor.* 4.

= the $\angle$BPD. *Theor.* 3

Thus AP + BP is a minimum, when AP, PB are *equally inclined* to CD.

EXAMPLE 4. *Given two intersecting straight lines* AB, AC, *and a point* P *between them; shew that of all straight lines which pass through* P *and are terminated by* AB, AC, *that which is bisected at* P *cuts off the triangle of minimum area.*



Let EF be that st. line, terminated by AB, AC, which is bisected at P.

Then the $\triangle$ FAE *is of minimum area.*

Proof. For let HK be *any* other st. line passing through P.

Through E draw EM par[l] to AC.

Then the $\triangle^s$ HPF, MPE are evidently congruent, *Theor.* 17.

and are therefore equal in area.

$\therefore$ the $\triangle$ HPF is less than the $\triangle$ KPE.

To each add the fig. AHPE;

then the $\triangle$ FAE is less than the $\triangle$ HAK.

That is, the $\triangle$ FAE is a minimum.

EXERCISES ON MAXIMA AND MINIMA.

1. Two sides of a triangle are given in length; how must they be placed in order that the area of the triangle may be a maximum?

Find the area of the greatest triangle in which $a=6{\cdot}8$ cm., and $b=4{\cdot}5$ cm.

2. Of all triangles of given base and area, shew that that which is isosceles has the least perimeter. [See Ex. 3, p. 316.]

Calculate the minimum perimeter of a triangle of which the base $=2{\cdot}0''$, and the area $=3{\cdot}12$ sq. in.

3. Construct a triangle of maximum area on a base of 10 cm., and having a vertical angle of 60°. Calculate its area.

4. With the origin as centre draw a circle of radius $1{\cdot}5''$, and draw AB joining the points $(3'', 0)$, $(0, 3'')$. Find a point in AB such that the tangents drawn from it to the circle contain the maximum angle. Measure the angle, and account for the result.

5. A straight rod slides between two straight rulers placed at right angles to one another; in what position is the triangle intercepted between the rulers and rod a maximum?

6. Divide a given straight line into two parts, so that the sum of the squares on the segments

(i) may be equal to a given square;

(ii) may be a minimum.

7. Through a point of intersection of two circles draw a straight line terminated by the circumferences,

(i) so that it may be of given length;

(ii) so that it may be a maximum.

8. Draw a circle to touch the axes of x and y at two points A and B, each $2''$ distant from the origin.

Find a point on the major arc AB such that the sum of its coordinates is a maximum.

Also find a point on the minor arc AB such that the sum of its coordinates is a minimum.

In each case calculate the sum, and test by measurement.

9. Straight lines are drawn from two given points to meet one another on the convex circumference of a given circle: prove that their sum is a minimum when they make equal angles with the tangent at the point of intersection.

10. Shew that of all triangles having a given vertical angle and altitude, that which is isosceles has the least area.

What is the least area a triangle can have if its vertical angle $=60°$, and altitude $=6$ cm.? Find its perimeter.

11. Given two intersecting tangents to a circle, draw a tangent to the convex arc so that the triangle formed by it and the given tangents may be of maximum area.

12. Find graphically (to the nearest degree) the greatest vertical angle which a triangle may have, when its base $=1 \cdot 6''$, and its area $=1 \cdot 2$ sq. in.

13. A and B are two points both within, or both without, a given circle. Find a point on the circumference at which AB subtends the greatest angle. [See Ex. 2, p. 315.]

14. A and B are two points on the x-axis distant $0 \cdot 8''$ and $1 \cdot 8''$ from the origin O. Find graphically, a point P on the y-axis, such that the angle APB is a maximum.

Calculate the length of OP, and measure the maximum angle.

15. A bridge consists of three arches, whose spans are 49 ft., 32 ft. and 49 ft. respectively: how far from the bridge is the point on either bank of the river at which the middle arch subtends the greatest angle?

16. From a given point P without a circle whose centre is C, draw a straight line to cut the circumference at A and B, so that the triangle ACB may be of maximum area.

Find the area of the greatest triangle that can be so drawn, when the radius $=6$ cm., and shew that the area is independent of the position of P.

17. Find the area of the greatest rectangle which can be inscribed in a circle of radius $5 \cdot 5$ cm.

18. A and B are two fixed points without a circle: find a point P on the circumference, such that AP^2+PB^2 may be a minimum.

[See Theor. 56.]

19. A segment of a circle is described on the chord AB: find a point C on its arc so that the sum of AC, BC may be a maximum.

20. *Of all triangles that can be inscribed in a circle that which has the greatest perimeter is equilateral.*

21. *Of all triangles that can be inscribed in a given circle that which has the greatest area is equilateral.*

22. *Of all triangles that can be inscribed in a given triangle that which has the least perimeter is the pedal triangle.*

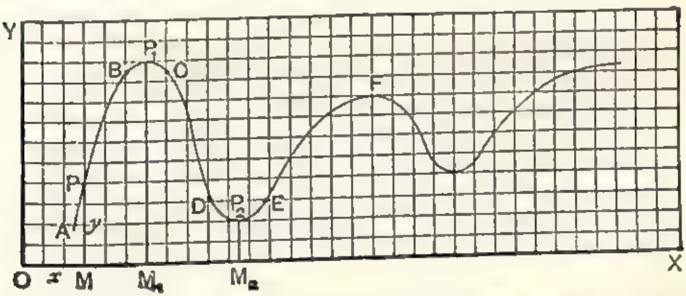
23. Of all rectangles of given area, the square has the least perimeter.

24. Describe the triangle of maximum area, having its angles equal to those of a given triangle, and its sides passing through three given points.

III. GRAPHS. APPLICATION TO MAXIMA AND MINIMA.

Problems dealing with the maximum or minimum values of some variable magnitude may often be conveniently treated by exhibiting its gradual changes by means of a graph. For details of graphical work the student may consult Hall's *Introduction to Graphical Algebra*. It will be sufficient here to explain the following general method of procedure. The variable magnitude whose values we have to examine may be denoted by y, and the quantity, in terms of which it is expressed, by x. By plotting a series of corresponding values of x and y on the coordinate axes OX, OY, a series of points is determined. If a continuous curve is drawn through them, the ordinate of each point denotes the value of the magnitude in question corresponding to a given value of the quantity x.

The advantage of this method is that it exhibits a visual picture of continuous change, so that the graph enables us to read off the value of y corresponding to *any* given value of x; and in particular the positions of maximum and minimum values are seen at a glance.



In this figure the continuous curve ABCDEF represents the graph of a variable quantity Q. As x increases gradually, the ordinate y travels parallel to OY, and its value at any point gives the value of Q for the corresponding value of x. At P_1 the value of y is greater than that at B or C on either side, and here Q is a maximum. Similarly at P_2 the value of y is less than that at D or E, and here Q is a minimum.

It will now be evident that maximum and minimum values occur at the *turning points* where the ordinates are algebraically greatest and least respectively in the immediate vicinity of such points.

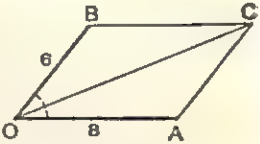
The following points should also be noticed:

(i) In any continuous curve maximum and minimum values occur alternately.

(ii) There will always be a maximum or a minimum value between any two equal values of the ordinate.

(iii) The slope of the curve at any point indicates the rate of change at that point of the quantity under discussion, and at each point of maximum or minimum value the tangent to the curve is parallel to the axis of x.

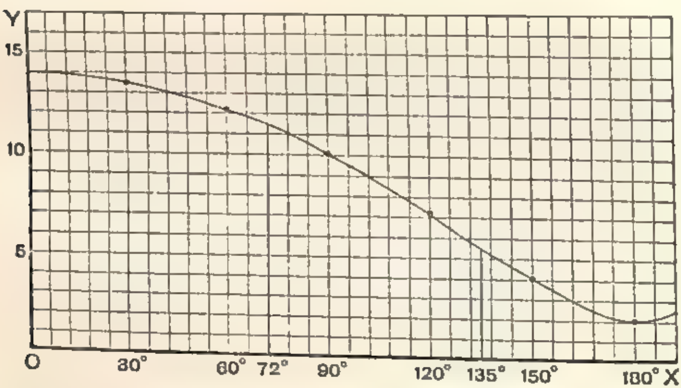
Example 1. OACB *is a parallelogram in which* OA = 8 cm. *and* OB = 6 cm. *If* OB *rotates about* O, *trace the changes in the value of* OC *as the angle* AOB *increases from* 0° *to* 180°. *Illustrate the changes by means of a graph. Read off the value of* OC *for an angle of* 72°, *and find the value of the angle when* OC = 5·6 cm.



By drawing a series of figures, increasing the angle AOB by increments of 30°, we shall find by measurement the corresponding values of OC and the ∠AOB to be as in the following table.

∠AOB	0°	30°	60°	90°	120°	150°	180°
OC	14·0	13·5	12·2	10·0	7·2	4·1	2·0

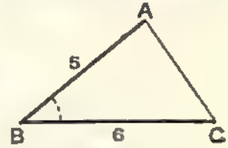
Denoting the angle by x, let its successive values be plotted on the x-axis, and let the corresponding values of OC be taken as ordinates. If each division on OX is taken to represent 6°, and each division on OY to represent 1 cm. we obtain the adjoining graph.



From this we see that when $x = 72°$, $y = 11·5$ cm., and when $y = 5·6$ cm. $x = 135°$.

The student should plot the graph for himself on a much larger scale than is possible on this page. He should also continue the values of OC for a few angles greater than 180°. He will then find that a minimum point has been reached when the angle AOB is 180°. Also it should be evident that at 360° the value of OC is again equal to 14, which is its maximum value.

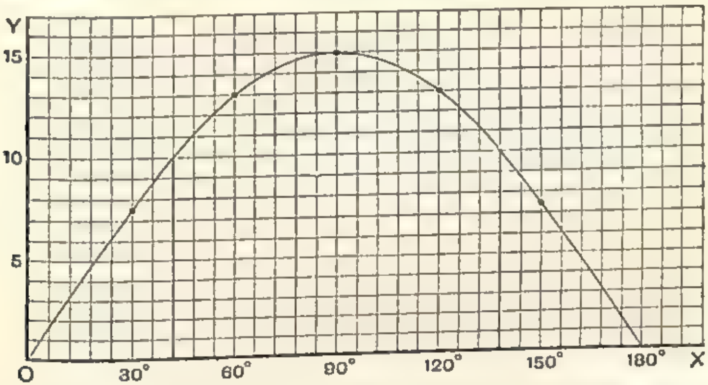
EXAMPLE 2. ABC *is a triangle in which* BC, BA *have constant lengths* 6 cm. and 5 cm. *If* BC *is fixed, and* BA *revolves about* B, *trace the changes in the area of the triangle as the angle* B *increases from* 0° *to* 180°. *Illustrate these changes by a graph, and determine for what values of the angle* B *the area is* 10 sq. cm. *Also find for what value of* B *the area is a maximum.*



Proceeding as in Ex. 5, p. 110, the corresponding values of the area and the angle will be as in the following table:

Angle	0°	30°	60°	90°	120°	150°	180°
Area in sq. cm.	0	7·5	13·0	15·0	13·0	7·5	0

Plot the values of the angle on the x-axis, and let the successive values of the area be taken as corresponding ordinates. Then with the same units as in the last example we obtain the adjoining graph.



It is easily seen that the maximum value of the ordinate is 15, corresponding to an angle of 90°.

Thus the area of the triangle is a maximum when the angle included by the given sides is a right angle.

Also the curve is symmetrical with regard to its maximum ordinate, so that there are two values of the angle which furnish a given area, other than the maximum. When the area is 10 sq. cm. the two values of the angle are 42° and 138°.

NOTE. The area of $\triangle ABC = \frac{1}{2}\, 5 \,.\, 6 \sin B = 15 \sin B$. Hence the graph may be plotted from a Table of sines. [See *Graphical Algebra*, p. 29.]

EXERCISES ON GRAPHS.

1. PQ is a perpendicular, 8 cm. in length, to a straight line XY, and PR is an oblique making an angle a with PQ. By giving to a the values 15°, 30°, 45°, 60°, 75°, find by measurement the corresponding values of PR, and tabulate the results. Illustrate the changes in PR by means of a graph, and find from it (i) the length of PR when $a=63°$, (ii) the value of a when PR = 8·8 cm.

2. In a triangle $a=4$ cm. $b=5$ cm. Plot a graph to shew the changes in the area of the triangle for different values of C. Find from the graph (i) the area of the triangle when C = 63°; (ii) the *values* of C when the area is 9·5 sq. cm.; (iii) how the sides must be placed when the area is a maximum.

3. A straight rod AB of length 5 cm. slides between two straight rulers CD, CE placed at right angles to each other. Draw a graph to shew the variations of the area of the triangle BCA for different values of the length CA.

Point out the position of AB when the area is a maximum.

4. AB is a straight line 10 cm. in length divided internally at P. As P moves from A to B illustrate graphically the variations of

(i) AP . PB; (ii) $AP^2 + PB^2$.

In each case determine from the graph the position of P which gives a maximum or minimum value.

5. In a triangle $c=6$ cm. and A = 60°. Trace the changes of a graphically for different values of b. Find from the graph the minimum value of a. Draw the triangle for this value, and hence check your result.

6. Through A, the extremity of the diameter AB of a semi-circle, a line AP is drawn to the circumference. Trace graphically the variations of the area of the triangle BAP for different values of the angle PAB. Find the value of this angle when the area is greatest.

7. By means of Theorem 73, shew that the graph of the equation $y=mx^2$, where m is constant, exhibits the changes in area of any series of similar rectilineal figures similarly placed on sides of varying length. Draw a graph to shew the changes in the area of a square as its side varies, and from the graph find approximately the side of a square whose area is 11·8 sq. in.

8. Draw the graphs of the curves represented by

$$\text{(i) } y=2x-\frac{x^2}{4}; \quad \text{(ii) } y=5-4x-x^2.$$

Find the maximum value of $5-4x-x^2$.

IV. HARMONIC SECTION.

DEFINITIONS.

1. Three quantities are said to be in **Arithmetical Progression** when the difference between the last pair is equal to that between the first pair.

Thus, a, b, c are in A.P. when

$$c-b=b-a,$$

and b is said to be an **Arithmetic Mean** between a and c.

2. Three quantities are said to be in **Geometrical Progression** when the ratio of the third to the second is the same as that of the second to the first.

Thus a, b, c are in G.P. when

$$\frac{c}{b}=\frac{b}{a},$$

and b is said to be a **Geometric Mean** between a and c.

3. Three quantities are said to be in **Harmonical Progression** when the first bears to the third the same ratio as the difference between the first and second bears to the difference between the second and third.

Thus a, b, c are in H.P. when

$$\frac{a}{c}=\frac{a-b}{b-c},$$

and b is said to be a **Harmonic Mean** between a and c.

NOTE. Since, by definition,

$$\frac{b-c}{c}=\frac{a-b}{a}, \quad \text{or} \quad \frac{b-c}{bc}=\frac{a-b}{ab},$$

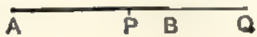
it follows that
$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a};$$

$\therefore$ the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., a result which is often useful.

4. If A, G, H denote the arithmetic, geometric, and harmonic means respectively between two given quantities a and b, it easily follows from the above definitions that

$$\mathrm{A}=\frac{a+b}{2}, \quad \mathrm{G}=\sqrt{ab}, \quad \mathrm{H}=\frac{2ab}{a+b}.$$

Definition. A finite straight line is said to be cut harmonically when it is divided internally and externally into segments which have the same ratio.



Thus AB is divided harmonically at P and Q, if

$$AP : PB = AQ : QB.$$

P and Q are said to be harmonic conjugates of A and B.

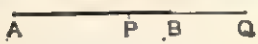
Now by taking the above proportion *alternately*,

we have $$PA : AQ = PB : BQ;$$

from which it is seen that if P and Q divide AB internally and externally in the same ratio, then A and B divide PQ externally and internally in the same ratio; hence A and B are harmonic conjugates of P and Q.

In other words: *if* AB *is divided harmonically at* P *and* Q, *then* PQ *is divided harmonically at* A *and* B.

Example 1. *If* AB *is divided internally at* P *and externally at* Q *in the same ratio, then* AB *is the harmonic mean between* AQ *and* AP.



For, *by hypothesis*, $AQ : QB = AP : PB$;

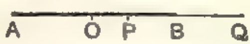
$\therefore$, *alternately*, $AQ : AP = QB : PB$,

that is, $AQ : AP = AQ - AB : AB - AP$;

$\therefore$ AQ, AB, AP are in Harmonic Progression.

Example 2. *If* AB *is divided harmonically at* P *and* Q, *and* O *is the middle point of* AB;

then $OP \cdot OQ = OA^2$.



For since AB is divided harmonically at P and Q,

$$\therefore\ AP : PB = AQ : QB;$$

$$\therefore\ AP - PB : AP + PB = AQ - QB : AQ + QB,$$

or, $$2OP : 2OA = 2OA : 2OQ;$$

$$\therefore\ OP \cdot OQ = OA^2.$$

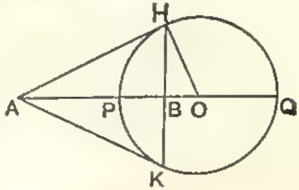
Conversely, if $OP \cdot OQ = OA^2$,

it may be shewn that

$$AP : PB = AQ : QB;$$

that is, that AB is divided harmonically at P and Q.

EXAMPLE 3. *The Arithmetic, Geometric, and Harmonic means of two straight lines may be thus represented graphically.*



Let AP, AQ be the given lines, whose Arithmetic, Geometric, and Harmonic Means are to be found.

On PQ as diameter draw a circle; and from A draw the tangents AH, AK.

Draw the chord of contact HK, cutting AQ at B.

Join OH.

Then (i) AO is the Arithmetic mean between AP and AQ: for clearly $AO = \frac{1}{2}(AP + AQ)$.

(ii) AH is the Geometric mean between AP and AQ:
for $AH^2 = AP \cdot AQ$. *Theor.* 58.

(iii) AB is the Harmonic mean between AP and AQ:
for, from the similar rt.-angled $\triangle^s$ AOH, HOB,

$$OA \cdot OB = OH^2 \qquad \textit{Theor. } 66, \text{ Cor.}$$
$$= OP^2.$$

$\therefore$ PQ is cut harmonically at A and B; Ex. 2. p. 324.

$\therefore$ also AB is cut harmonically at P and Q.

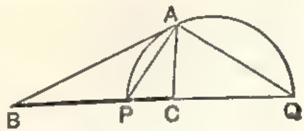
That is, AB is the Harmonic mean between AP and AQ.

And from the similar rt.-angled triangles OAH, HAB,

$$AO \cdot AB = AH^2; \qquad \textit{Theor. } 66, \text{ Cor.}$$

$\therefore$ *the Geometric mean between two straight lines is the mean proportional between their Arithmetic and Harmonic means.*

EXAMPLE 4. *Given the base of a triangle and the ratio of the other sides, to find the locus of the vertex.*



Let BC be the given base, and let BAC be any triangle standing upon it, such that BA : AC = the given ratio.

It is required to find the locus of A.

Bisect the $\angle$BAC internally and externally by AP, AQ.

Then BC is divided internally at P, and externally at Q,
so that BP : PC = BQ : QC = the given ratio;
$\therefore$ P and Q are fixed points.

And since AP, AQ bisect the $\angle$BAC internally and externally,
$\therefore$ the $\angle$PAQ is a rt. angle;
$\therefore$ the locus of A is the circle described on PQ as diameter.

EXERCISES ON HARMONIC SECTION.

1. If AB is divided harmonically at X and Y, shew that

$$\text{(i)}\quad \frac{2}{AB}=\frac{1}{AX}+\frac{1}{AY}.$$

$$\text{(ii)}\quad \frac{2}{XY}=\frac{1}{BY}+\frac{1}{AY}.$$

2. X and Y are harmonic conjugates of A and B;

(i) if $AB=2{\cdot}4''$, and $AX=1{\cdot}5''$, find AY;

(ii) if $XY=1{\cdot}5$ cm., and $AY=2$ cm., find BY.

3. Any straight line is cut harmonically by the arms of an angle and its internal and external bisectors.

4. Given three points B, P, C in a straight line: find the locus of points at which BP and PC subtend equal angles.

5. If through the middle point of the base of a triangle any line is drawn intersecting one side of the triangle, the other produced, and the line drawn parallel to the base from the vertex, it is divided harmonically.

6. If from either base angle of a triangle a line is drawn intersecting the median from the vertex, the opposite side, and the line drawn parallel to the base from the vertex, it is divided harmonically.

7. P, Q are harmonic conjugates of A and B, and C is an external point; if the angle PCQ is a right angle, shew that CP, CQ are the internal and external bisectors of the angle ACB.

8. AB is a given straight line, bisected at O, and divided harmonically at X and Y.

Trace the change of position of Y as X moves from O to B.

Taking AB equal to 20 cm. draw a graph to illustrate the variations of OY as OX changes.

9. Justify the following construction for finding the harmonic mean between two straight lines of given length.

Let AB and CD be the given lines, and let them be placed so as to be parallel. Join their ends towards the same parts by AC and BD, and towards opposite parts by AD and BC, cutting at O. Then if POQ is drawn parallel to the given lines and terminated by AC and BD, PQ is the required harmonic mean.

DEFINITIONS.

1. A series of points in a straight line is called a **range**. If the range consists of four points, of which one pair are harmonic conjugates with respect to the other pair, it is said to be a **harmonic range**.

2. A series of straight lines drawn through a point is called a **pencil**.

The point of concurrence is called the **vertex** of the pencil, and each of the straight lines is called a **ray**.

A pencil of four rays drawn from any point to a harmonic range is said to be a **harmonic pencil**.

3. A straight line drawn to cut a system of lines is called a **transversal**.

4. A system of four straight lines, no three of which are concurrent, is called a **complete quadrilateral**.

These straight lines will intersect two and two in *six* points, called the **vertices** of the quadrilateral; and each of the *three* straight lines which join the opposite vertices is called a **diagonal**.

THEOREMS ON HARMONIC SECTION.

1. *If a transversal is drawn parallel to one ray of a harmonic pencil, the other three rays intercept equal parts upon it: and conversely.*

2. *Any transversal is cut harmonically by the rays of a harmonic pencil.*

3. *In a harmonic pencil, if one ray bisect the angle between the other pair of rays, it is perpendicular to its conjugate ray. Conversely, if one pair of rays form a right angle, then they bisect internally and externally the angle between the other pair.*

4. *If* A, P, B, Q *and* a, p, b, q *are harmonic ranges, one on each of two given straight lines, and if* Aa, Pp, Bb, *the straight lines which join three pairs of corresponding points, meet at* S; *then will* Qq *also pass through* S.

5. *If two straight lines intersect at* A, *and if* A, P, B, Q *and* A, p, b, q *are two harmonic ranges one on each straight line (the points corresponding as indicated by the letters), then* Pp, Bb, Qq *will be concurrent: also* Pq, Bb, Qp *will be concurrent.*

6. *Use the last result to prove that in a complete quadrilateral each diagonal is cut harmonically by the other two.*

V. CENTRES OF SIMILITUDE.

EXAMPLE 1. *In two circles if any two parallel radii are drawn (one in each circle), the straight line joining their extremities cuts the line of centres in one or other of two fixed points.*

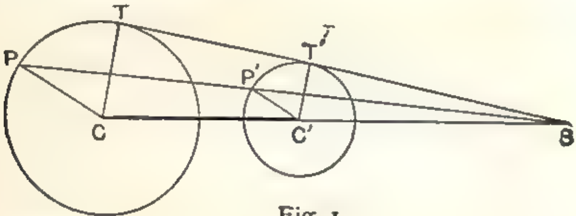


Fig. 1.

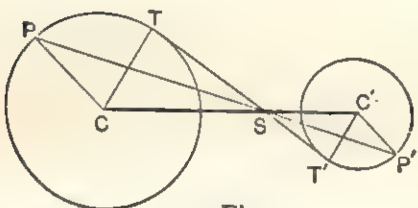


Fig. 2.

Take two circles whose centres are C and C′, and radii r and r' respectively; and let CP, C′P′ be any two par^{l} radii drawn in the *same sense* in Fig. 1, and in *opposite senses* in Fig. 2. Let PP′ cut CC′ at S.

It is required to prove that (whatever be the direction of CP, CP′) S *is in one or other of two fixed positions.*

Proof. In both Figs. the △s SCP, SC′P′ are equiangular;

$$\therefore \text{SC} : \text{SC}' = \text{CP} : \text{C}'\text{P}'$$
$$= r : r'.$$

Hence S divides CC′ $\left\{\begin{array}{l}\text{externally in Fig. 1}\\ \text{internally in Fig. 2}\end{array}\right\}$ in the fixed ratio $r : r'$.

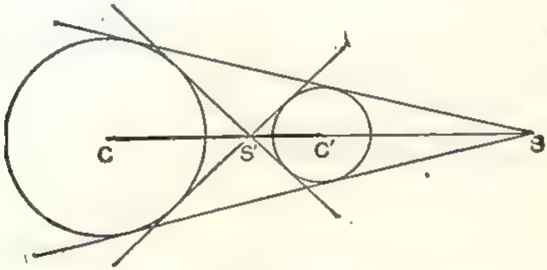
∴ in each Fig., S is a *fixed point* for all directions of CP, C′P′.

COROLLARY. Let TT′ be a common tangent to the two circles, *direct* in Fig. 1, and *transverse* in Fig. 2.

Then in both cases the radii CT, C′T′ are par[l];

∴ TT′ cuts the line of centres at S.

DEFINITION. In the figure given below the points S and S′ which divide the line of centres of two circles externally and internally in the ratio of their radii are called Centres of Similitude, the former being the centre of *direct* and the latter of *transverse* similitude.



COROLLARY. Since $\frac{SC}{SC'} = \frac{r}{r'} = \frac{S'C}{S'C'}$,

the centres of the circles and the centres of similitude form an harmonic range.

Hence the transverse and direct common tangents intersect on the line of centres at points which divide that line harmonically.

EXERCISES.

1. From centres C and C′, 5·5 cm. apart, draw two circles of radii 3·2 cm. and 1·2 cm. respectively, and determine (i) graphically (ii) by calculation the distances of their centres of similitude from C.

2. Two circles whose centres are C and C′ respectively have radii 1·8″ and 1·0″, and their direct centre of similitude is 2·7″ distant from C. Find the distance (i) between their centres (ii) between their centres of similitude.

3. In Fig. 1 of the preceding page, if SP cuts the circles (C) and (C′) again at Q and Q′ respectively, shew that

$$SQ \cdot SP' = SP \cdot SQ' = ST \cdot ST'.$$

EXERCISES.

(*On Centres of Similitude. Continued.*)

4. In the triangle ABC, I is the in-centre, and I_1 the ex-centre opposite to A. If AI_1 cuts BC at Y, shew that A and Y are the centres of similitude of the two circles.

5. *Shew that the orthocentre and centroid of a triangle are respectively the external and internal centres of similitude of the circumscribed and nine-points circle.*

6. *If a variable circle touches two fixed circles, the line joining the points of contact passes through a centre of similitude. Distinguish between the different cases.*

7. *Describe a circle which shall touch two given circles and pass through a given point.*

8. *Describe a circle which shall touch three given circles.*

9. C_1, C_2, C_3 *are the centres of three given circles;* S'_1, S_1, *are the internal and external centres of similitude of the pair of circles whose centres are* C_2, C_3, *and* S'_2, S_2, S'_3, S_3, *have similar meanings with regard to the other two pairs of circles: shew that*

(i) S'_1C_1, S'_2C_2, S'_3C_3 *are concurrent;*

(ii) *the six points* S_1, S_2, S_3, S_1', S'_2, S'_3, *lie three and three on four straight lines.* [See Theorems IX. and X., pp. 344, 345.]

ORTHOGONAL CIRCLES.

DEFINITION. Circles which intersect at a point, so that the two tangents at that point are at right angles to one another, are said to be orthogonal, or to cut one another orthogonally.

1. If two circles cut one another orthogonally, the tangent to each circle at a point of intersection will pass through the centre of the other circle.

2. If two circles cut one another orthogonally, the square on the distance between their centres is equal to the sum of the squares on their radii.

3. Find the locus of the centres of all circles which cut a given circle orthogonally at a given point.

4. Describe a circle to pass through a given point and cut a given circle orthogonally at a given point.

VI. POLE AND POLAR.

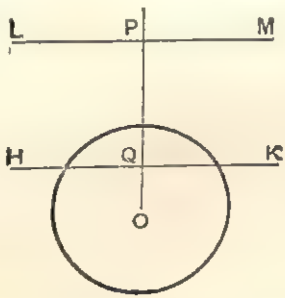
DEFINITIONS.

1. If in any straight line drawn from the centre of a circle two points are taken such that the rectangle contained by their distances from the centre is equal to the square on the radius, each point is said to be the **inverse** of the other.

Thus in the figure given below, if O is the centre of the circle, and if $OP \cdot OQ = (\text{radius})^2$, then each of the points P and Q is the inverse of the other.

It is clear that if one of these points is within the circle the other must be without it.

2. The **polar** of a given point with respect to a given circle is the straight line drawn through the inverse of the given point at right angles to the line which joins the given point to the centre: and with reference to the polar the given point is called the **pole**.



Thus in the adjoining figure, if $OP \cdot OQ = (\text{radius})^2$, and if through P and Q, LM and HK are drawn perp. to OP; then HK is the polar of the point P, and P is the pole of the st. line HK with respect to the given circle; also LM is the polar of the point Q, and Q the pole of LM.

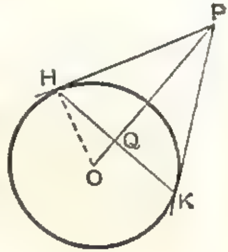
It is clear that the polar of an *external* point must intersect the circle, and that the polar of an *internal* point must fall without it: also that the polar of a point *on the circumference* is the tangent at that point.

EXAMPLE 1. *The polar of an external point with reference to a circle is the chord of contact of tangents drawn from the given point to the circle.*

From the external point P let two tangents PH, PK be drawn to a circle of which O is the centre.

Join HK.

It is required to prove that HK *is the polar of* P.



Now OP evidently cuts the chord of contact HK at right angles at Q.

Join OH.

Then from the similar rt.-angled $\triangle^s$ POH, HOQ,

$$OP : OH = OH : OQ;$$

$$\therefore OP \cdot OQ = (\text{radius})^2;$$

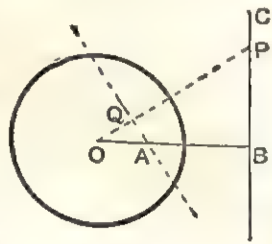
hence HK is the polar of P.

Q.E.D.

EXAMPLE 2. *If* A *and* P *are any two points, and if the polar of* A *with respect to any circle passes through* P, *then the polar of* P *must pass through* A.

Let BC be the polar of the point A with respect to a circle whose centre is O, and let BC pass through P.

It is required to prove that the polar of P *passes through* A.



Join OP; and from A draw AQ perp. to OP. We shall shew that AQ is the polar of P.

Now since BC is the polar of A,

∴ the ∠ABP is a rt. angle; *Def.* 2, page 331,

and the ∠AQP is a rt. angle: *Constr.*

∴ the four points A, B, P, Q are concyclic;

$$\therefore OQ \cdot OP = OA \cdot OB \qquad \textit{Theor. 58.}$$

$$= (\text{radius})^2, \text{ for CB is the polar of A.}$$

And since AQ is perp. to OP,

∴ AQ is the polar of P.

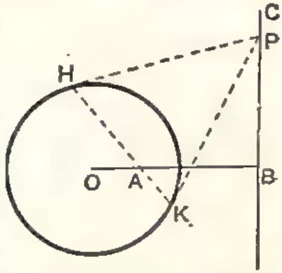
That is, the polar of P passes through A.

Q.E.D.

NOTE. A similar proof applies to the case when the given point A is without the circle, and the polar BC cuts it.

The above Theorem is known as the **Reciprocal Property** of Pole and Polar.

EXAMPLE 3. *The locus of the intersection of tangents drawn to a circle at the extremities of all chords which pass through a given point within the circle is the polar of that point.*



Let A be the given point within the circle. Let HK be *any* chord passing through A; and let the tangents at H and K intersect at P.

It is required to prove that the locus of P *is the polar of the point* A.

(α) To shew that P lies on the polar of A.

Since HK is the chord of contact of tangents drawn from P,

$\therefore$ HK is the polar of P. Ex. 1, p. 332.

But HK, the polar of P, passes through A;

$\therefore$ the polar of A passes through P: Ex. 2, p. 332.

that is, the point P lies on the polar of A.

(β) To shew that *any* point on the polar of A satisfies the given conditions.

Let BC be the polar of A, and let P be any point on it.

Draw tangents PH, PK, and let HK be the chord of contact.

Now from Ex. 1, p. 332, we know that the chord of contact HK is the polar of P,

and we also know that the polar of P must pass through A; for P is on BC, the polar of A: Ex. 2, p. 332.

that is, HK passes through A.

$\therefore$ P is the point of intersection of tangents drawn at the extremities of a chord passing through A.

From (α) and (β) we see that the required locus is the polar of A.

NOTE. If A is outside the circle the theorem (α) still holds good; but the converse theorem (β) is not true for all points in BC. For if A is without the circle, the polar BC will intersect it; and no point on that part of the polar which is within the circle can be a point of intersection of tangents.

We now see that

(i) *The Polar of an* external *point with respect to a circle is the chord of contact of tangents drawn from it.*

(ii) *The Polar of an* internal *point is the locus of the intersections of tangents drawn at the extremities of all chords which pass through it.*

(iii) *The Polar of a* point on the circumference *is the tangent at that point.*

EXAMPLE 4. *Any chord of a circle through a fixed point* P *is divided harmonically by* P *and the polar of* P.

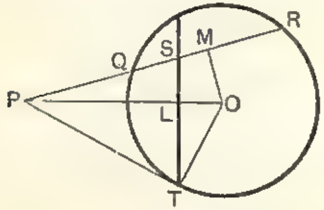


Fig. 1.

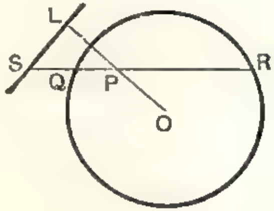


Fig. 2.

Let O be the centre of the given circle, and let QR be a chord passing through the given point P.

(i) When P is an external point (Fig. 1).

Draw a tangent PT, and let the polar of P cut PO in L, and QR in S.

It is required to prove that QR *is divided harmonically at* P *and* S.

Draw OM perp. to QR, and join OT.

Then $$PQ \cdot PR = PT^2$$
$$= PL \cdot PO, \text{ since PTO is a rt. } \angle,$$
$$= PM \cdot PS, \text{ since S, L, O, M are concyclic.}$$
$$\therefore\ 2PQ \cdot PR = 2PM \cdot PS$$
$$= (PQ + PR)PS; \qquad \text{Ex. 9, p. 65.}$$
$$\therefore\ PS = \frac{2PQ \cdot PR}{PQ + PR};$$

∴ PQ, PS, PR are in Harmonical Progression;
that is, PS is divided harmonically at Q and R,
∴ also QR is divided harmonically at P and S.

(ii) When P is an internal point (Fig. 2).

Let SL be the polar of P.

Then since the polar of P passes through S, the polar of S passes through P.

∴ by the former case QR is divided harmonically at S and P.

The above theorem is known as the **Harmonic Property of Pole and Polar.**

DEFINITION.

A triangle so related to a circle that each side is the polar of the opposite vertex is said to be **self-conjugate** with respect to the circle.

EXERCISES ON POLE AND POLAR.

1. *The straight line which joins any two points is the polar with respect to a given circle of the point of intersection of their polars.*

2. *The point of intersection of any two straight lines is the pole of the straight line which joins their poles.*

3. *Find the locus of the poles of all straight lines which pass through a given point.*

4. *Find the locus of the poles, with respect to a given circle, of tangents drawn to a concentric circle.*

5. *If two circles cut one another orthogonally and* PQ *be any diameter of one of them; shew that the polar of* P *with regard to the other circle passes through* Q.

6. *If two circles cut one another orthogonally, the centre of each circle is the pole of their common chord with respect to the other circle.*

7. *Any two points subtend at the centre of a circle an angle equal to one of the angles formed by the polars of the given points.*

8. O *is the centre of a given circle, and* AB *a fixed straight line.*
P *is any point in* AB; *find the locus of the point inverse to* P *with respect to the circle.*

9. *Given a circle, and a fixed point* O *on its circumference:* P *is any point on the circle: find the locus of the point inverse to* P *with respect to any circle whose centre is* O.

10. *Given two points* A *and* B, *and a circle whose centre is* O; *shew that the rectangle contained by* OA *and the perpendicular from* B *on the polar of* A *is equal to the rectangle contained by* OB *and the perpendicular from* A *on the polar of* B.

11. *Four points* A, B, C, D *are taken in order on the circumference of a circle;* DA, CB *intersect at* P, AC, BD *at* Q, *and* BA, CD *in* R: *shew that the triangle* PQR *is self-conjugate with respect to the circle.*

12. *Give a* linear *construction for finding the polar of a given point with respect to a given circle. Hence find a linear construction for drawing a tangent to a circle from an external point.*

13. *If a triangle is self-conjugate with respect to a circle, the centre of the circle is at the orthocentre of the triangle.*

14. *The polars, with respect to a given circle, of the four points of a harmonic range form a harmonic pencil: and conversely.*

VII. THE RADICAL AXIS.

EXAMPLE 1. *To find the locus of points from which the tangents drawn to two given circles are equal.*

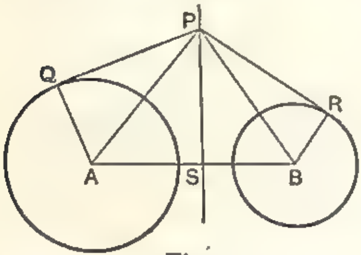


Fig. 1.

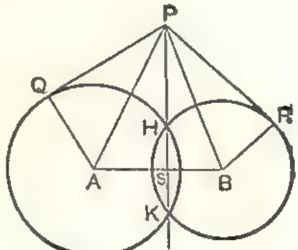


Fig. 2.

Let A and B be the centres of the given circles, whose radii are a and b; and let P be any point such that the tangent PQ drawn to the circle (A) is equal to the tangent PR drawn to the circle (B).

It is required to find the locus of P.

Join PA, PB, AQ, BR, AB;
from P draw PS perp. to AB.

Then because $PQ = PR$, $\therefore$ $PQ^2 = PR^2$.

But $PQ^2 = PA^2 - AQ^2$; and $PR^2 = PB^2 - BR^2$: *Theor.* 29.

$\therefore$ $PA^2 - AQ^2 = PB^2 - BR^2$;

that is, $PS^2 + AS^2 - a^2 = PS^2 + SB^2 - b^2$; *Theor.* 29.

or, $AS^2 - a^2 = SB^2 - b^2$.

Hence AB is divided at S, so that $AS^2 - SB^2 = a^2 - b^2$:

$\therefore$ S is a *fixed point*.

Hence all points from which equal tangents can be drawn to the two circles lie on the straight line which cuts AB at rt. angles, so that the difference of the squares on the segments of AB is equal to the difference of the squares on the radii.

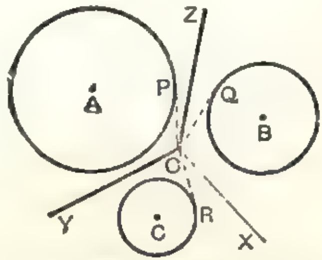
Again, by simply retracing these steps, it may be shewn that in Fig. 1 every point in SP, and in Fig. 2 every point in SP exterior to the circles, is such that tangents drawn from it to the two circles are equal.

Hence we conclude that in Fig. 1 the whole line SP is the required locus, and in Fig. 2 that part of SP which is without the circles.

In either case SP is said to be the **Radical Axis** of the two circles.

COROLLARY. *If the circles cut one another as in Fig. 2, it is clear that the Radical Axis is identical with the straight line which passes through the points of intersection of the circles;* for it follows readily from Theorem 58 that tangents drawn to two intersecting circles from any point in the common chord produced are equal.

EXAMPLE 2. *The Radical Axes of three circles taken in pairs are concurrent.*



Let there be three circles whose centres are A, B, C.

Let OZ be the radical axis of the ⊙s (A) and (B);

and OY the Radical Axis of the ⊙s (A) and (C), O being the point of their intersection.

It is required to prove that the radical axis of the ⊙s (B) *and* (C) *passes through* O.

It will be found that the point O is either *without* or *within all the circles.*

I. When O is without the circles.

From O draw OP, OQ, OR tangents to the ⊙s (A), (B), (C).

Then because O is a point on the radical axis of (A) and (B);

$$\therefore \text{OP} = \text{OQ}.$$

And because O is a point on the radical axis of (A) and (C),

$$\therefore \text{OP} = \text{OR};$$

$$\therefore \text{OQ} = \text{OR};$$

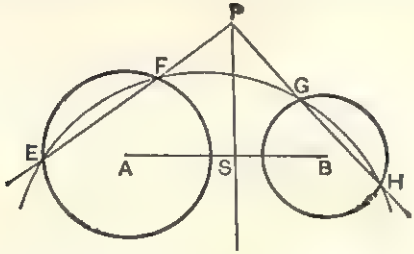
∴ O is a point on the radical axis of (B) and (C);

that is, the radical axis of (B) and (C) passes through O.

II. If the circles intersect in such a way that O is within them all; the radical axes are then the common chords of the three circles taken two and two; and it is required to prove that these common chords are concurrent. This may be shewn indirectly by Theorem 57.

DEFINITION. The point of intersection of the radical axes of three circles taken in pairs is called the **radical centre.**

EXAMPLE 3. *To draw the radical axis of two given circles.*



Let A and B be the centres of the given circles.

It is required to draw their radical axis.

(α) If the given circles intersect, then the st. line drawn through their points of intersection will be the radical axis.

(β) But if the given circles do not intersect,

describe any circle so as to cut them in E, F and G, H.

Join EF and HG, and produce them to meet in P.

Join AB; and from P draw PS perp. to AB.

Then PS *is the radical axis of the* ⊙s (A), (B).

Proof. From the ⊙ EFGH, PE . PF = PH . PG.

Now the sq. on the tangent from P to the ⊙ (A) = PE . PF;

and the sq. on the tangent from P to the ⊙ (B) = PH . PG.

Hence the tangents from P to the ⊙s (A) and (B) are equal;

∴ P is a point on the radical axis.

And since PS is perp. to the line of centres,

∴ PS is the radical axis. *Ex.* 1, p. 336.

DEFINITION. If each pair of circles in a given system have the same radical axis, the circles are said to be co-axal.

EXERCISES ON THE RADICAL AXIS.

1. *Shew that the radical axis of two circles bisects any one of their common tangents.*

2. *If tangents are drawn to two circles from any point on their radical axis; shew that a circle described with this point as centre and any one of the tangents as radius, cuts both the given circles orthogonally.* [See Def. p. 330.]

3. O *is the radical centre of three circles, and from* O *a tangent* OT *is drawn to any one of them: shew that a circle whose centre is* O *and radius* OT *cuts all the given circles orthogonally.*

4. *If three circles touch one another, taken two and two, shew that their common tangents at the points of contact are concurrent.*

5. *If circles are described on the three sides of a triangle as diameter, their radical centre is the orthocentre of the triangle.*

6. *All circles which pass through a fixed point and cut a given circle orthogonally, pass through a second fixed point.*

7. *Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.*

8. *Describe a circle to pass through two given points and cut a given circle orthogonally.*

9. *Find the locus of the centres of all circles which cut two given circles orthogonally.*

10. *Describe a circle to pass through a given point and cut two given circles orthogonally.*

11. *The difference of the squares on the tangents drawn from any point to two circles is equal to twice the rectangle contained by the straight line joining their centres and the perpendicular from the given point on their radical axis.*

12. *In a system of co-axal circles which do not intersect, any point is taken on the radical axis; shew that a circle described from this point as centre, with radius equal to the tangent drawn from it to any one of the circles, will meet the line of centres in two* fixed *points.*

[*These fixed points are called the* **Limiting Points** *of the system.*]

13. *In a system of co-axal circles the two limiting points and the points in which any one circle of the system cuts the line of centres form a harmonic range.*

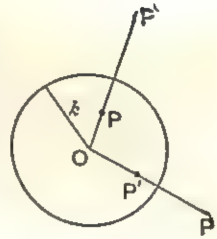
14. *In a system of co-axal circles a limiting point has the same polar with regard to all the circles of the system.*

15. *If two circles are orthogonal any diameter of one is cut harmonically by the other.*

VIII. INVERSION.

DEFINITIONS.

1. If from any fixed point O a straight line OP is drawn, and a point P′ is taken on OP, or OP produced, such that OP . OP′ $= k^2$, where k is constant, then each of the points P and P′ is said to be the inverse of the other with respect to the circle whose centre is O and radius k.



2. The point S is called the origin of inversion, and k is called the radius of inversion. Also k^2 is sometimes referred to as the constant of inversion.

3. If P traces out a locus, to every position of P there is a corresponding position of P′. The locus of P′ is called the inverse of the locus of P.

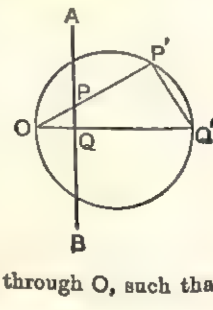
From Definition 1 it is clear that any straight line passing through the origin is its own inverse.

EXAMPLE 1. *To find the inverse of a straight line not passing through the origin of inversion.*

Let P be any point on the given st. line AB, O the origin, and k the radius of inversion.

Draw OQ perp. to the given line. Take P′ and Q′ the inverses of P and Q.

Join P′Q′.



Then OP . OP′ $= k^2$

= OQ . OQ′.

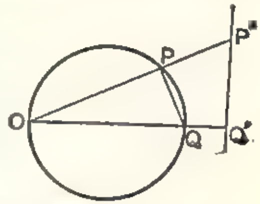
∴ the pts. P, P′, Q, Q′ are concyclic;

∴ the ∠OP′Q′ = the ∠OQP

= a rt. ∠.

Hence the locus of P′ is a circle which passes through O, such that the diameter OQ′ is perp. to the given line.

EXAMPLE 2. *To find the inverse of a circle with respect to a point in its circumference.*



Let OQ be the diameter of the given circle which passes through the origin O.

Take any point P on this circle; and with k as radius of inversion, let Q′ and P′ be the inverses of Q and P.

Join PQ, P′Q′.

Then $\quad OP \cdot OP' = k^2$

$= OQ \cdot OQ'$.

∴ the pts. P, P′, Q, Q′ are concyclic;

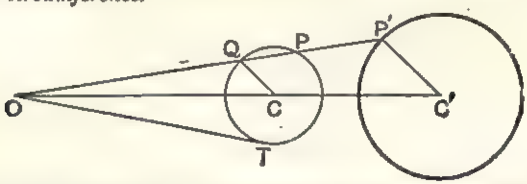
∴ the ∠OQ′P′ = the ∠OPQ

= a rt. ∠.

∴ P′Q′ is perp. to OQ′.

Hence the locus of P′ is a st. line perp. to the diameter through the origin.

EXAMPLE 3. *To find the inverse of a circle with respect to a point not on the circumference.*



Let O be the origin and P any point on the given circle whose centre is C.

Let P′ be the inverse of P, so that $OP \cdot OP' = k^2$.

Let OP meet the given circle again in Q. Join QC.

Draw OT a tangent to the circle, and let $OT = t$.

Then $\quad OP \cdot OP' = k^2$, and $OP \cdot OQ = t^2$.

$$\therefore \frac{OP \cdot OP'}{OP \cdot OQ} = \frac{k^2}{t^2};$$

$$\therefore OP : OQ = k^2 : t^2.$$

Draw P′C′ parˡ to QC to meet OC produced in C′.

Then $\quad OC' : OC = OP' : OQ$

$= k^2 : t^2$.

∴ C′ is a fixed point.

Also $\quad C'P' : CQ = OP' : OQ$

$= k^2 : t^2$.

∴ C′P′ is constant, and the locus of P′ is a circle whose centre is C′.

COROLLARY. The origin is a centre of similitude of the circle and its inverse.

EXAMPLE 4. *Any line through the origin of inversion cuts two inverse loci at the same angle on opposite sides of the line.*

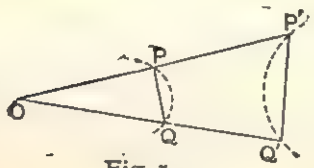


Fig. 1.

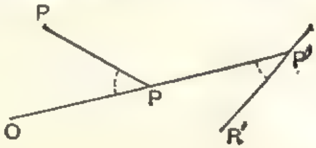


Fig. 2.

Let P and Q be two points on a locus, and let P′, Q′ be their inverses with respect to O.

Then $$OP \, . \, OP' = k^2$$
$$= OQ \, . \, OQ'.$$

$\therefore$ the pts. P, P′, Q, Q′ are concyclic;

$\therefore$ $\angle OQP =$ the $\angle OP'Q'$.

Now let Q move up to P, so that the st. line QP ultimately becomes the tangent at P to the locus of P. Then at the same time the st. line Q′P′ becomes the tangent at P′ to the locus of P′.

Hence in Fig. 2, if PR and P′R′ are the tangents at P and P′,

the $\angle OPR =$ the $\angle OP'R'$,

that is, OPP′ cuts the loci of P and P′ at the same angle *on opposite sides of* OPP′.

COROLLARY. *At any point of intersection two curves cut at the same angle as their inverses at the inverse point. Also if two curves touch at* P *their inverses touch at the inverse point* P′.

EXAMPLE 5. *To express the distance between two points in terms of the distance between their inverses and the distances of these points from the origin.*

If P′, Q′ are the inverses of P, Q, [Fig. 1 of Example 4.]

$$OP \, . \, OP' = k^2 = OQ \, . \, OQ';$$

and from the similar triangles OPQ, OQ′P′,

$$\frac{P'Q'}{PQ} = \frac{OP'}{OQ} = \frac{OP \, . \, OP'}{OP \, . \, OQ} = \frac{k^2}{OP \, . \, OQ}.$$

$$\therefore \; P'Q' = \frac{k^2 \, . \, PQ}{OP \, . \, OQ}.$$

EXERCISES ON INVERSION.

1. If O, P, Q, R are collinear points and P′, Q′, R′ the inverses of P, Q, R with respect to O, prove that

   (i) if OP, OQ, OR are in Arithmetical Progression, then OP′, OQ′, OR′ are in Harmonical Progression.

   (ii) if OP, OQ, OR are in Geometrical Progression, then OP′, OQ′, OR′ are also in Geometrical Progression.

2. Find the inverse of the circum-circle of an isosceles triangle with respect to the vertex of the triangle as origin.

3. Shew that a circle can be inverted into itself with respect to any point O as origin.

   [Take k equal to the length of the tangent from O.]

4. Shew that a circle inverts into itself with respect to the centre of any orthogonal circle.

5. AB is a chord of a circle bisected at O. Shew that, with O as origin, and OA as radius of inversion, the circle inverts into itself.

6. Shew that any two circles can be inverted into themselves.

   [See Ex. 1. p. 336. Take the origin O on SP, and take k equal to the length of the tangent from O.]

7. Shew that any three circles can be inverted into themselves.

   [See Ex. 2. p. 337.]

8. Shew that if a straight line cuts a circle, each may be inverted into the other by suitable selection of the origin and constant of inversion.

9. Shew that any three circles may be inverted into three circles whose centres are collinear.

   [See Ex. 3, p. 339, and take the origin of inversion on the orthogonal circle.]

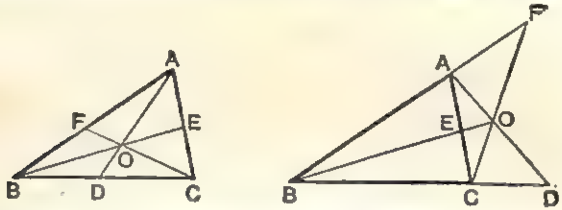
10. Shew that the diameters of a circle may be inverted into a series of co-axal circles orthogonal to the inverse of the given circle.

11. P, Q, R are three points taken in order on a straight line. Find the inverse of the statement

$$PQ+QR=PR.$$

IX. CEVA'S THEOREM.

If three concurrent straight lines are drawn from the angular points of a triangle to meet the opposite sides, then the product of three alternate segments taken in order is equal to the product of the other three segments.



Let AD, BE, CF be drawn from the vertices of the $\triangle$ ABC to intersect at O, and cut the opposite sides at D, E, F.

It is required to prove that

$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

Now the $\triangle^s$ AOB, AOC have a common base AO; and it may be shewn, by drawing perpendiculars from B and C to AD, that

$$BD : DC = \text{the alt. of } \triangle\, AOB : \text{the alt. of } \triangle\, AOC;$$

$$\therefore \frac{BD}{DC} = \frac{\triangle\, AOB}{\triangle\, AOC};$$

similarly,
$$\frac{CE}{EA} = \frac{\triangle\, BOC}{\triangle\, BOA};$$

and
$$\frac{AF}{FB} = \frac{\triangle\, COA}{\triangle\, COB}.$$

Multiplying these ratios, we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1;$$

or,
$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

NOTE. The converse of this theorem, which may be proved indirectly, is very important; it may be enunciated thus:

If three straight lines drawn from the vertices of a triangle cut the opposite sides so that the product of three alternate segments taken in order is equal to the product of the other three, then the three straight lines are concurrent.

That is, if $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$,

then AD, BE, CF *are concurrent.*

X. MENELAUS' THEOREM.

If a transversal is drawn to cut the sides, or the sides produced, of a triangle, the product of three alternate segments taken in order is equal to the product of the other three segments.

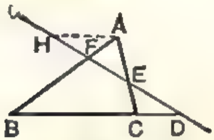


Fig. 1.

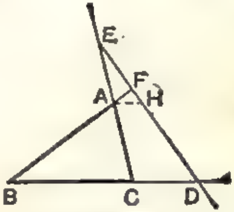


Fig. 2.

Let ABC be a triangle, and let a transversal meet the sides BC, CA, AB, or these sides produced, at D, E, F.

It is required to prove that

$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

Draw AH par^l to BC, meeting the transversal at H.

Then from the similar $\triangle^s$ DFB, HAF,

$$\frac{AF}{FB} = \frac{AH}{BD};$$

and from the similar $\triangle^s$ DCE, HAE,

$$\frac{CE}{EA} = \frac{CD}{AH}.$$

$\therefore$, by multiplication, $\frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{CD}{BD};$

that is, $\frac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB} = 1,$

or, $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$

NOTE. In this theorem the transversal must either meet two sides and the third side produced, as in Fig. 1; or all three sides produced, as in Fig. 2.

The converse of this theorem may be proved indirectly:

If three points are taken in two sides of a triangle and the third side produced, or in all three sides produced, so that the product of three alternate segments taken in order is equal to the product of the other three segments, the three points are collinear.

DEFINITIONS.

1. If two triangles are such that the three straight lines joining corresponding vertices are concurrent, they are said to be co-polar.

2. If two triangles are such that the three points of intersection of corresponding sides are collinear, they are said to be co-axial.

EXERCISES.

1. By means of Ceva's Theorem prove the following properties of a triangle.

(i) The perpendiculars to the sides from their middle points are concurrent.

(ii) The bisectors of the angles are concurrent.

(iii) The medians are concurrent.

2. D, E, F are the points of contact of the in-circle of a triangle with the sides BC, CA, AB respectively. If EF, FD, DE meet these sides respectively in P, Q, R, shew that P, Q, R are collinear.

3. With the same notation as in Example 2, shew that the points B, D, C, P form a harmonic range.

4. If the tangents at A, B, C of the circum-circle of the triangle ABC meet the opposite sides in D, E, F respectively, shew that

$$BD : CD = BA^2 : AC^2.$$

Hence prove that D, E, F are collinear.

5. *The straight lines which join the vertices of a triangle to the points of contact of the inscribed circle (or any of the three escribed circles) are concurrent.*

6. *The middle points of the diagonals of a complete quadrilateral are collinear.* [See Def. 4, p. 327.]

7. *Shew that each diagonal of a complete quadrilateral is divided harmonically by the other two diagonals.*

8. *Co-polar triangles are also co-axial; and conversely co-axial triangles are also co-polar.*

9. *The six centres of similitude of three circles lie three by three on four straight lines.*

PART VI.

SOLID GEOMETRY.

LINES AND PLANES.

DEFINITIONS AND FIRST PRINCIPLES.

1. From the Definitions of Part I. it will be remembered that

(i) A point has *no magnitude*; that is to say neither length, breadth, nor thickness.

(ii) A line has *length*, without breadth or thickness.

(iii) A surface has *length* and *breadth*, without thickness.

(iv) A solid has *length*, *breadth*, and *thickness*.

Thus a point is said to be of no dimension;
a line .... of one dimension;
a surface .... of two dimensions;
a solid .... of three dimensions.

2. Solids, surfaces, lines, and points are thus related to one another:

(i) Solids are bounded by surfaces.

(ii) Surfaces are bounded by lines; and surfaces meet in lines.

(iii) Lines are bounded by points; and lines meet in points.

It must also be noticed that a line which intersects a surface cuts it in one or more points.

3. A **plane** is a surface such, that if any two points are taken in it, the straight line joining them lies wholly in the surface.

Unless the contrary is stated, the straight lines treated in this Section are supposed to be of infinite length, and the planes of infinite extent.

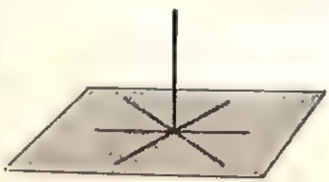
4. Lines which are drawn on a plane, or through which a plane may be made to pass, are said to be **co-planar.**

5. Lines through which a plane cannot be made to pass are said to be **skew.**

6. Planes are said to be **parallel** when they do not meet though indefinitely extended.

7. A straight line and a plane are said to be **parallel** when they do not meet though both are indefinitely extended.

8. A straight line is **perpendicular to a plane** when it is perpendicular to *every* straight line which meets it in that plane. Such a straight line is said to be **normal to the plane.**



AXIOMS.

1. *A straight line drawn through two points on a plane must, if indefinitely produced, lie wholly in that plane.*

2. *Through a given straight line, or through two given points, an infinite number of planes may pass*; for we may suppose a plane to turn about any straight line that lies in it, and thus to pass in succession through an infinite number of positions.

3. *If a plane of unlimited extent turns about a straight line that lies in it, it may be made to pass through any point in space outside the given line.*

From these principles it follows that a straight line may be related to a plane in three ways:

(i) It may be *parallel to the plane*, in which case it has *no* point in common with the plane.

(ii) It may *cut the plane*, in which case it has *one* (and only one) point in common with the plane.

(iii) It may *lie in the plane*, in which case it has an indefinite number of points in common with the plane.

Again, two straight lines may be related to one another in three ways:

If the lines are co-planar, they may

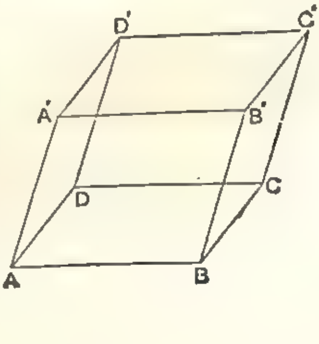
(i) intersect one another,

or (ii) be parallel.

If the lines are not co-planar, that is, if they are skew,

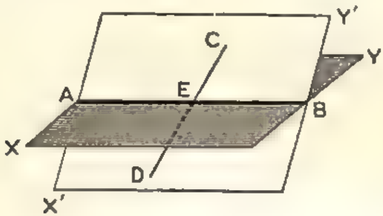
(iii) they neither intersect nor are parallel.

For example, in the solid figure represented in the margin, (i) the edges AB, BC, in the plane ABCD, *intersect*; (ii) the edges AB, DC in the same plane *are parallel*; (iii) the edges AB, A′D′, through which no plane can be made to pass, are skew, and *neither intersect nor are parallel*.



THEOREM 79. [Euclid XI. 2.]

One, and only one, plane may be made to pass through any two intersecting straight lines.



Let the two given st. lines AB, CD intersect at E.

It is required to prove that one, and only one, plane may be made to pass through AB, CD.

Proof. Take any plane XY passing through AEB, and let this plane turn about AB until it passes through C, taking the position X′Y′. The position of the revolving plane is then *fixed*, so that only one plane can pass through the st. line AB and the point C.

And since E and C are points in this plane, the whole st. line CED lies in it;

∴ one, and only one, plane can pass through AB and CD.

Q.E.D.

COROLLARY. *Any three straight lines, of which each pair cut one another, must be co-planar.*

INFERENCES.

Hence the position of a plane is fixed,

(i) if it passes through a given straight line and a given point outside that line;

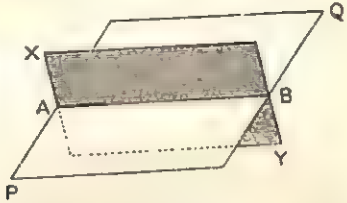
(ii) if it passes through two intersecting straight lines;

(iii) if it passes through three points not collinear;

(iv) if it passes through two parallel straight lines.

THEOREM 80. [Euclid XI. 3.]

Two intersecting planes cut one another in a straight line, and in no point outside it.



It is required to prove that the two intersecting planes PQ, XY *cut in a straight line, and in no point outside it.*

Proof. Let A and B be points in both the planes PQ, XY; then the *straight* line joining A and B lies wholly in both planes; that is, the planes intersect along the st. line AB.

And since the planes pass through the st. line AB, they can have no point in common outside AB; for otherwise they would coincide. Q.E.D.

NOTE. It will now be seen that (i) if three or more concurrent straight lines cut a given straight line, they are co-planar;

(ii) if three or more parallel straight lines cut a given straight line, they are co-planar.

The generation of a plane. Thus a plane may be generated by

(i) a straight line turning about a fixed point and sliding over a fixed straight line;

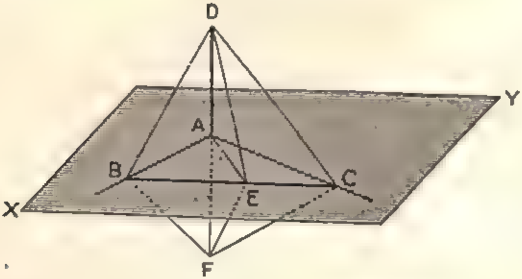
(ii) a straight line sliding over two fixed intersecting straight lines, or two fixed parallel lines;

(iii) a straight line moving parallel to itself and sliding over a fixed straight line.

Triangles and Quadrilaterals in space. The sides of every *triangle* must lie in one plane (Theor. 79); but the sides of a *quadrilateral* need not lie in one plane, as may be seen by folding a plane quadrilateral about either diagonal. A quadrilateral so formed is called skew or gauche, two adjacent sides lying in one plane and the other two in another plane.

THEOREM 81. [Euclid XI. 4.]

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.



Let AD be perp. to each of the st. lines AB, AC.

It is required to prove that AD *is perp. to the plane* XY *which passes through* AB, AC.

In the plane XY draw *any* line AE through A; and in the same plane draw BC to cut AB, AE, AC at the points B, E, C.

Produce DA to F, making AF equal to AD.

Join DB, DE, DC; and FB, FE, FC.

Proof. In the $\triangle^s$ BAD, BAF,

because BA bisects DF at rt. angles,

$\therefore$ BD = BF. *Theor.* 12. *Cor.* 2.

Similarly CD = CF.

Hence if the $\triangle$ BFC is turned about its base BC, until the vertex F comes into the plane of the $\triangle$ BDC, then F must coincide with D, for the $\triangle^s$ BFC, BDC are identically equal.

$\therefore$ EF coincides with ED; that is, EF = ED.

Hence in the $\triangle^s$ DAE, FAE,

since DA, AE, ED = FA, AE, EF respectively,

$\therefore$ the $\angle$DAE = the $\angle$FAE;

$\therefore$ DA is perp. to any line AE which meets it in the plane XY;

that is, DA is perp. to the planes of AB, AC.

Q.E.D.

QUESTIONS AND EXERCISES.

1. "*Straight lines are parallel when they do not meet though indefinitely produced.*" Supply what is wanting to this definition, justifying your answer by an illustration.

2. Give instances from the walls and edges of a room of

   (i) parallel planes;
   (ii) lines parallel to a plane;
   (iii) lines perpendicular to a plane;
   (iv) pairs of skew lines.

3. "*Surfaces cut in lines.*" Need these lines be straight? Give instances to the contrary from any surfaces known to you.

4. How many straight lines may be perpendicular to a given straight line at a given point

   (i) in space of *two* dimensions?
   (ii) in space of *three* dimensions?

5. Place two pencils so as to illustrate the fact that a straight line is not necessarily perpendicular to a plane, if it is perpendicular to *one* line in the plane.

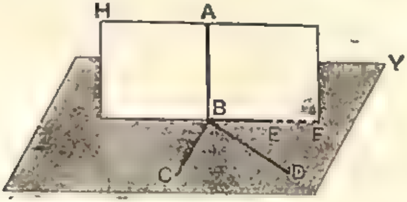
6. Shew that three straight lines drawn from a point in space may be so placed that each is perpendicular to the other two.

   Prove that in this case each line is perpendicular to the plane of the other two, and illustrate from the walls and edges of a room.

7. From O the centre of a circle a perpendicular OA is erected to the plane of the circle. Shew that all points on the circumference are equidistant from A.

THEOREM 82. [Euclid XI. 5.]

All straight lines drawn perpendicular to a given straight line at a given point are co-planar.



Let each of the st. lines BC, BD, BE be perp. to the st. line AB at the point B.

It is required to prove that BC, BD, BE *are in one plane.*

Proof. Let XY be the plane which passes through BC, BD; and let HF be the plane which passes through AB, BE.

Suppose the planes HF, XY to intersect in the st. line BF.

Because AB is perp. to BC and BD,

∴ AB is also perp. to BF which meets it in the plane of BC, BD; *Theor.* 81.

Hence the $\angle^s$ ABE, ABF are both right angles and in the same plane HF;

∴ BE coincides with BF.

That is, BC, BD, BE are in the same plane XY.

Q.E.D.

COROLLARY. *If a right angle revolves about one of its arms, the other arm generates a plane.*

HYPOTHETICAL CONSTRUCTION. *Through any point in a straight line a plane may be supposed to be constructed perpendicular to the given line.*

DEFINITIONS.

1. The direction of a plumb-line hanging freely at rest is said to be **vertical**.

2. Any plane perpendicular to a vertical line is said to be **horizontal**.

3. Any straight line drawn in a horizontal plane is said to be **horizontal**.

QUESTIONS AND EXERCISES.

1. How many vertical straight lines can pass through a given point, and how many horizontal lines?

2. A sheet of note-paper partly opened out is placed with two of its short edges on a horizontal table; why is the crease vertical?

3. Shew that two observations with a spirit-level are sufficient to determine if a plane is horizontal, provided that the two positions are not parallel.

4. A circle of radius 4·2 cm. is drawn on a horizontal plane, and a straight line OP of length 5·6 cm. stands vertically at the centre O. Find the distance of P from any point on the circumference, shewing that the distance is the same for all such points.

5. Two straight lines AB, CD bisect one another at O, and OP is perpendicular to them both; shew that

$$\text{(i) } PA = PB; \qquad \text{(ii) } PC = PD;$$

and that this result is independent of the angle at which AB and CD intersect.

If $$AB = 3{\cdot}6'', \quad CD = 1{\cdot}4'', \text{ and } OP = 2{\cdot}4'',$$

find PA and PC.

6. ABCD is a horizontal square, and at O its mid-point (that is, the intersection of its diagonals) a vertical rod OP is fixed, the extremity P being joined by threads to the angular points of the square.

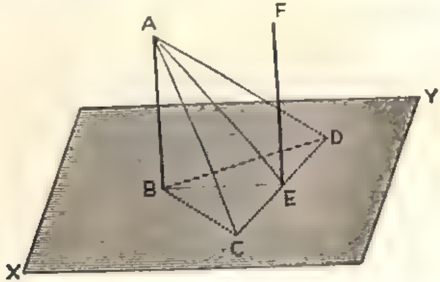
(i) Prove that PA, PB, PC, PD are all equal.

(ii) If each side of the square is 20 cm., and the height of the rod is 40 cm., find PA to the nearest millimetre.

(iii) If PA = 85 cm., and the height of the rod is 75 cm., find a side of the square to the nearest millimetre.

THEOREM 83. [Euclid XI. 8.]

If two straight lines are parallel, and if one of them is perpendicular to a plane, then the other is also perpendicular to the same plane.



Let AB, FE be two par[l] st. lines, cutting the plane XY at B and E; and let AB be perp. to the plane.

It is required to prove that FE *is also perp. to the plane* XY.

Join AE, BE; and through E draw CED in the plane XY perp. to BE, making EC and ED of any equal lengths.

Join BC, BD; also AC, AD.

Proof. Since EB bisects CD at right angles,

∴ BC = BD. *Theor.* 12. *Cor.* 2.

And in the △[s] ABC, ABD,
since BC = BD, AB is common, and ∠ABC = ∠ABD,
for AB is perp. to the plane of BC, BD,

∴ AC = AD. *Theor.* 4.

Again, in the △[s] CEA, DEA,
since CE = DE, EA is common, and AC = AD,
∴ the ∠CEA = the ∠DEA;
that is, CE is perp. to EA.

But CE is perp. to EB by construction;
∴ CE is perp. to the plane containing EA and EB.

And EF is in this plane; for both EA, EB lie in the plane of the par[ls] AB, FE.

∴ CE is also perp. to EF.

Again since AB, FE are par[l], and since by hypothesis the ∠ABE is a rt. angle,

∴ the ∠FEB is a rt. angle. *Theor.* 14.

Thus FE, being perp. both to EB and EC, is also perp. to the plane XY which contains them.

Q.E.D.

Conversely, *if* AB *and* FE *are both perpendicular to the plane* XY, *they are parallel to one another.*

With the same construction, prove, as before, that CE is perp. to the plane of EA, EB.

Now it follows from the hypothesis that CE is perp. to EF;

∴ EF is in the plane of EA, EB.

But AB is also in the plane of EA, EB;

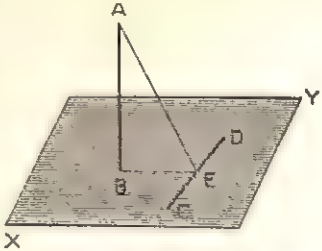
∴ AB, FE are *co-planar.*

And since, by hypothesis, each of the ∠[s] ABE, FEB is a rt. angle,

∴ AB is par[l] to FE.

Q.E.D.

Corollary. *If* AB *is perpendicular to a plane* XY, *and if from* B, *the foot of the perpendicular, a line* BE *is drawn perpendicular to any line* CD *in the plane, then the join* AE *is also perpendicular to* CD.



Make EC, ED of any equal length.

Join BC, BD; also AC, AD.

The proof follows as in the last Theorem.

This important result is known as "*The Theorem of the Three Perpendiculars.*"

THEOREM 84.

Through any point in or outside a plane there can always be one, and only one, straight line perpendicular to the plane.

(i) Let A be the given point in the plane XY.

Take BC any st. line through A in the given plane.

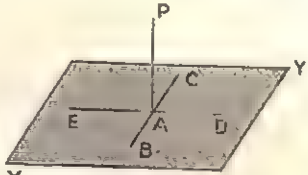


Fig. 1

Let AP, making a rt. angle with BC, revolve about BC as axis. Then AP traces out a plane perp. to BC. Let this plane cut the plane XY in the st. line DAE.

Now as AP turns from AD to AE, it must pass through one position in which it is perp. to DE, *i.e.* perp. both to BC and DE, and consequently perp. to the plane XY.

(ii) Let A be the given point outside the plane XY.

Take BC any st. line in the plane; and let AD be the perp. from A on BC.

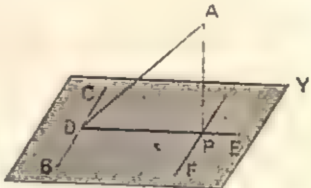


Fig. 2

Draw DE perp. to BC in the plane XY.

Let AP be the perp. from A on DE.

Then AP *must be perp. to the plane* XY.

Proof. Through P draw PF par^l to BC.

Now BC, being perp. to DA and DE, is perp. to the plane ADE;

hence FP is also perp. to the plane ADE. *Theor.* 83.

$\therefore$ the $\angle$ APF is a rt. angle,

that is, AP is perp. both to PF and DE, and consequently to the plane XY.

(iii) *There can be only one perpendicular to the plane* XY *from the point* P, *whether this point is in the plane or outside it.*

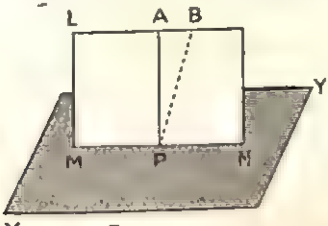


Fig. 1

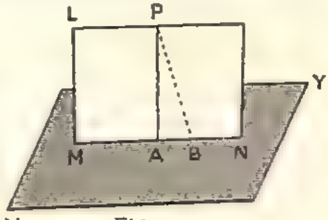


Fig. 2

For, if possible, suppose two perp^s PA, PB to be drawn from P to the plane XY, and let the plane through PA, PB cut the plane XY in the straight line MN.

Then PA, PB are both perp. to MN and in the same plane with it, which is impossible.

Q.E.D.

EXERCISES.

1. Given two set-squares, shew how a straight rod may be placed perpendicular to a plane at a given point in it.

2. Shew how, with a set-square and straight rod, the construction of Fig. 2. p. 358 may be practically applied to finding the position of a straight line perpendicular to a plane XY from a point A outside it.

3. BC is a straight line drawn on a plane surface of a solid, and A an external point [see Fig. 2, p. 358]: it is required to find the position of a line through A perpendicular to BC. Why is the usual method (Problem 4, p. 74) here inapplicable as a *practical* construction?

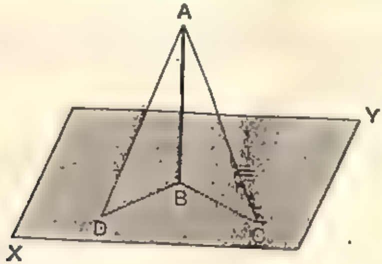
Devise a feasible construction, with ruler and compasses only, for finding the foot D of the required perpendicular AD. Hence shew how a perpendicular may be drawn from A to the plane.

[For another construction of a line perpendicular to a plane from a point outside it, see p. 361, Ex. 3.]

THEOREM 85.

(i) *Of all straight lines drawn from an external point to a plane, the perpendicular is the shortest.*

(ii) *Of* obliques *drawn from the given point, those which cut the plane at equal distances from the foot of the perpendicular are equal.*



(i) Let AB be the perpendicular, and AC any oblique, drawn from the external point A to the plane XY.

It is required to shew that AB *is less than* AC.

Join BC.

Proof. Since AB is perp. to the plane XY, it is also perp. to BC which meets it in that plane.

Hence in the $\triangle$ ABC, the $\angle$ ACB is less than the $\angle$ ABC;

$\therefore$ AB is less than AC. *Theor.* 10.

(ii) Let the obliques AC, AD cut the plane XY at equal distances BC, BD from the foot of the perpendicular AB.

It is required to prove that AC = AD.

Proof. Since AB is perp. to the plane XY, it is also perp. to BC, BD which meet it in that plane.

Hence the $\triangle$s ABC, ABD are congruent;

for AB is common, BC = BD, and $\angle$ ABC = $\angle$ ABD,

$\therefore$ AC = AD.

Q.E.D.

EXERCISES.

1. Find the locus of the feet of equal obliques drawn from an external point to a plane.

2. P is a point outside the plane of the triangle ABC and equidistant from its vertices. If the perpendicular from P to the plane cuts it at S, shew that S is the circum-centre of the triangle ABC.

If the $\triangle$ ABC is rt. angled at C, and $a=4{\cdot}8''$, $b=3{\cdot}6''$, and SP$=4{\cdot}0''$; find PA.

3. Deduce from the last example a practical method of drawing a perpendicular from a given external point to a plane, having given ruler and compasses, and a straight rod of any length greater than the required perpendicular.

4. Give a geometrical construction for drawing a straight line equally inclined to three straight lines which meet in a point, but are not in the same plane.

5. AB is a straight line in a plane XY, and PQ is the perpendicular to the plane from a point P outside it.

(i) If QR is perpendicular to AB, shew that PR is also perpendicular to AB.

(ii) If PR is perpendicular to AB, shew that QR is also perpendicular to AB.

6. ABCD is a square on a side of 0·96 metre, and from Q its mid-point a rod QP of length 1·40 m. is fixed perpendicular to the plane. If R is the mid-point of the side BC, calculate the value of cos PRQ correct to the third decimal figure.

7. AB is the line of section of two planes, and from P any point in AB two lines PQ, PR are drawn perpendicular to AB, one in each plane. Shew that the line drawn from any point in PQ perpendicular to the plane containing it lies in the plane of PQ, PR.

8. Prove that :

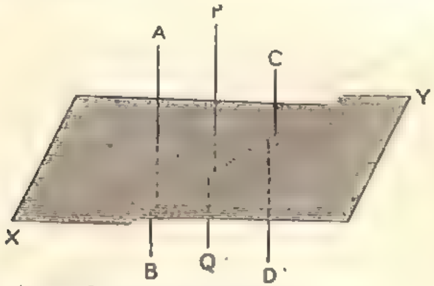
(i) All points in space equidistant from *two* given points lie in a *plane.*

(ii) All points in space equidistant from *three* non-collinear points lie in a *straight line* (viz. the intersection of two planes).

(iii) There is only one point equidistant from *four* non-co-planar points (viz. the intersection of two straight lines).

THEOREM 86. [Euclid XI. 9.]

Straight lines which are parallel to a given straight line are parallel to one another.



Let AB, CD be each par^l to the st. line PQ.

It is required to prove that AB, CD *are par^l to one another.*

Proof. Suppose XY to be any plane perp. to PQ.

Then because AB is par^l to PQ,
$\therefore$ AB is perp. to the plane XY. *Theor.* 83.
And because CD is par^l to PQ,
$\therefore$ CD is perp. to the plane XY.

$\therefore$ AB and CD, being both perp. to the plane XY are par^l to one another.
Theor. 83. *Converse.*

Q.E.D.

NOTE. This Theorem has already been proved [Page 40] in the special case where AB, CD, and PQ are *co-planar.*

EXERCISES.

1. AB, CD, EF are three equal and parallel straight lines not in one plane: shew that the triangles ACE, BDF are congruent.

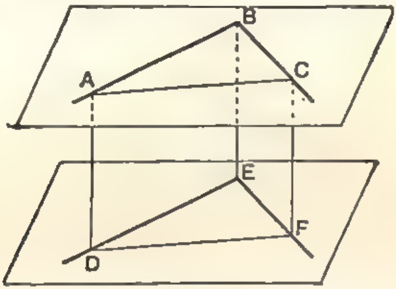
2. If the middle points of adjacent sides of a skew quadrilateral are joined, prove that the figure so formed is a parallelogram.

3. If a triangle revolves about its base, shew that the vertex describes a circle.

4. At O the mid-point of a regular hexagon drawn on a horizontal plane, a normal OP of length 9·6 cm. is erected, and one side AB is bisected at X. If AB = 4·0 cm., find the values of PA, OX, PX, cos OAP, cos OXP, and shew that AB is perpendicular to PX.

THEOREM 87. [Euclid XI. 10.]

If two intersecting straight lines are respectively parallel to two other intersecting straight lines not in the same plane with them, then the first pair and the second pair contain equal angles.



Let the st. lines AB, BC be respectively par[l] to the st. lines DE, EF, which are not in the same plane with them.

It is required to prove that the $\angle$ABC = *the* $\angle$DEF.

Make BA equal to ED; and make BC equal to EF.

Join AD, BE, CF, AC, DF.

Proof. Because BA is equal and par[l] to ED,

$\therefore$ AD is equal and par[l] to BE. *Theor.* 20.

Similarly CF is equal and par[l] to BE.

Hence AD and CF, being each equal and par[l] to BE, are equal and par[l] to one another; *Theor.* 86.

$\therefore$ AC is equal and par[l] to DF.

Then in the $\triangle$[s] ABC, DEF,

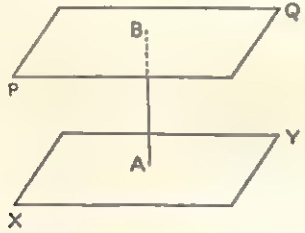
because AB, BC, AC = DE, EF, DF, respectively,

$\therefore$ the $\angle$ABC = the $\angle$DEF. *Theor.* 7.

Q.E.D.

Theorem 88. [Euclid XI. 14.]

Planes to which the same straight line is perpendicular are parallel to one another.



Let the straight line AB be perp. to each of the planes XY, PQ.

It is required to prove that the planes XY, PQ *are parallel.*

Proof. Since AB is perp. to the plane XY, it is perp. to the line joining A to *any* point in that plane.

Similarly AB is perp. to the line joining B to *any* point in the plane PQ.

Hence if the planes XY, PQ had a point in common, then, by joining this point to A and B, two perpendiculars could be drawn from it to AB, one in each plane; which is impossible.

Thus the planes XY, PQ have no point in common, and are therefore parallel.

Q.E.D.

EXERCISES.

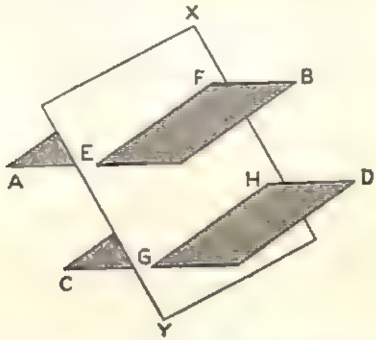
1. AB, CD are normal to a plane, cutting it at B and D If AB, CD are equal in length and on the same side of the plane, show that ABDC is a rectangle.

2 Use the last Example to find the locus of points equidistant from a given plane.

3. Find the locus of points equidistant from two given points.

Theorem 89. [Euclid XI. 16.]

If two parallel planes are cut by a third plane, their lines of section with it are parallel.



Let the par' planes AB, CD be cut by the plane XY in the lines of section EF, GH.

It is required to prove that EF, GH *are parallel.*

Proof. Now EF and GH cannot meet, since they lie respectively in the planes AB, CD, which have no point in common.

Moreover EF and GH are co-planar, since they are in the given plane XY.

∴ EF and GH are parallel. Q.E.D.

EXERCISES.

1. Through a given point there can only be one plane parallel to a given plane.

2. Prove that a straight line which is normal to one of two parallel planes is also normal to the other.

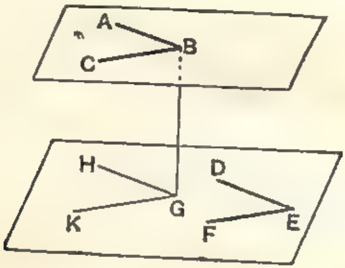
3. Shew that planes which are parallel to the same plane are parallel to one another.

4. Prove that intercepts of parallel lines between parallel planes are equal.

5. Given two pairs of parallel planes, how many lines of section will they have? Shew that these lines are parallel.

Theorem 90. [Euclid XI. 15.]

If two intersecting straight lines are parallel respectively to two other intersecting straight lines which are not in the same plane with them, then the plane containing the first pair is parallel to the plane containing the second pair.



Let the st. lines AB, BC be respectively par^l to the st. lines DE, EF, which are not in the same plane with them.

It is required to prove that the plane of AB, BC *is parallel to the plane of* DE, EF.

From B let BG be drawn perp. to the plane of DE, EF, and meeting it at G.

Draw GH, GK par^l respectively to ED, EF.

Proof. Because BG is perp. to the plane of DE, EF,
$\therefore$ each of the $\angle^s$ BGH, BGK is a rt. angle.

Now *by hypothesis* BA is par^l to ED,
and *by construction* GH is par^l to ED;
$\therefore$ BA is par^l to GH. *Theor.* 86.

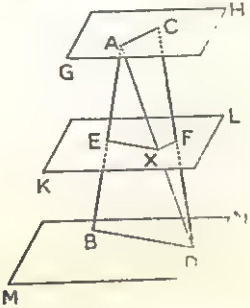
And since the $\angle$ BGH is a rt. angle;
$\therefore$ the $\angle$ ABG is a rt. angle.
Similarly the $\angle$ CBG is a rt. angle.
$\therefore$ BG is perp. to the plane of AB, BC.

And, by construction, BG is perp. to the plane of ED, EF;
$\therefore$ these planes are par^l. *Theor.* 88.

Q.E.D.

Theorem 91. [Euclid XI. 17.]

Straight lines which are cut by parallel planes are cut proportionally.



Let the st. lines AB, CD be cut by the three par[l] planes GH, KL, MN at the points A, E, B, and C, F, D.

It is required to prove that AE : EB = CF : FD.

Join AC, BD, AD; and let AD meet the plane KL at the point X: join EX, XF.

Proof. Because the two par[l] planes, KL, MN are cut by the plane ABD,

∴ the lines of section EX, BD are par[l]. *Theor.* 89.

And because the two par[l] planes GH, KL are cut by the plane DAC,

∴ the lines of section XF, AC are par[l].

Now since EX is par[l] to BD, a side of the △ABD,

∴ AE : EB = AX : XD. *Theor.* 60.

Again because XF is par[l] to AC, a side of the △DAC,

∴ AX : XD = CF : FD.

Hence AE : EB = CF : FD.

Q.E.D.

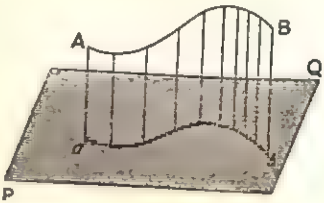
EXERCISES.

1. If two intersecting planes are cut by two parallel planes, the lines of section of the first pair with each of the second pair contain equal angles. Point out an exceptional case.

2. Shew how to determine in a given straight line the point which is equidistant from two fixed points. When is this impossible?

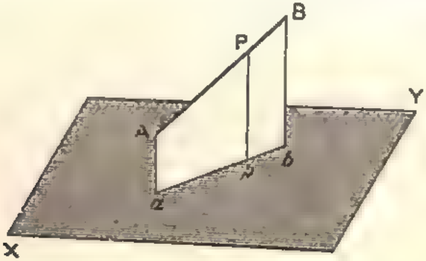
DEFINITION. The projection of a line on a plane is the locus of the feet of perpendiculars drawn from all points in the given line to the plane.

Thus in the adjoining figure the projection of the line AB on the plane PQ is the line *ab*.



THEOREM 92.

The projection of a straight line on a plane is itself a straight line.



Let AB be the given st. line, and XY the given plane; and from P, *any* point in AB, let P*p* be drawn perp. to the plane XY and meeting it at *p*.

It is required to shew that the locus of p *is a straight line.*

Suppose A*a* and B*b* to be the perps. from A and B to the plane XY.

Proof. Now A*a*, P*p*, B*b*, being all perp. to the plane XY, are par^l to one another. *Theor.* 83, *Converse.*

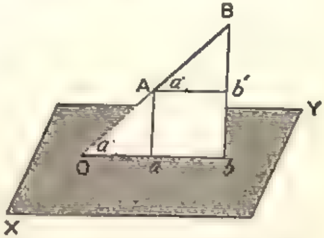
And since these par^ls all cut AB, they are co-planar.

∴ the point *p* is in the line of section of the planes A*b*, XY; that is, *p* is in the st. line *ab*.

But *p* is *any* point in the projection of AB,

∴ the projection of AB is the st. line *ab*. Q.E.D.

COROLLARY 1. *The angle which a straight line* AB *makes with a plane* XY *may be measured by the angle between* AB *and its projection* ab *on the plane*; for a straight line and its projection are co-planar.



Thus, if AB and ab, produced if necessary, meet at O, the angle between AB and the plane XY is measured by the angle B$\text{O}b$.

COROLLARY 2. *To find the length of the projection of a straight line* AB *on a plane* XY *in terms of* AB *and the angle which it makes with the plane.*

Let α be the angle which AB makes with the plane XY.

Draw $\text{A}b'$ par[l] to ab, cutting $\text{B}b$ at b'.

Then the $\angle \text{BA}b'$ = the corresponding $\angle \text{BO}b = \alpha$.

Now from the rt. angled $\triangle \text{BA}b'$, $\dfrac{\text{A}b'}{\text{AB}} = \cos\alpha$;

hence $ab = \text{A}b' = \text{AB}\cos\alpha$.

[See p. 272, Ex. (ii).]

NOTE. Since as α increases from 0 to 90°, $\cos\alpha$ decreases, it follows that as the inclination of AB to the plane increases, the projection ab decreases.

EXERCISES.

1. If a straight line is parallel to a plane, shew that it is parallel to its projection on that plane.

2. Compare the length of AB with that of its projection on the plane XY, when AB

   (i) is parallel to the plane;

   (ii) is perpendicular to the plane;

   (iii) is inclined to the plane at an angle of 60°.

3. Shew that equal obliques drawn to a plane from an external point have equal projections on that plane.

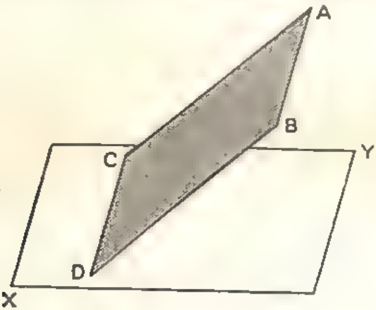
4. Prove that parallel straight lines have parallel projections on a plane. Is there any exception to this?

5. If A′B′, C′D′ are the projections of two parallel straight lines AB, CD on any plane, shew that

$$\text{AB} : \text{CD} = \text{A}'\text{B}' : \text{C}'\text{D}'.$$

Theorem 93.

If a straight line outside a given plane is parallel to any straight line drawn on the plane, it is also parallel to the plane itself.



Let AB be par[l] to CD drawn on the plane XY.

It is required to prove that AB *is par[l] to the plane* XY.

Proof. Suppose AD to be the plane of the par[ls] AB, CD; so that CD is the line of section of the planes AD, XY.

Then AB, being in the plane AD, must meet the plane XY, if at all, at some point in CD.

But, by hypothesis, AB can never meet CD;

∴ AB can never meet the plane XY, and is consequently par[l] to it.

Q.E.D.

Conversely. *If a straight line is parallel to a plane, then any plane passing through the given line and intersecting the given plane, will cut it in a line parallel to the given line.*

In the above Figure let the st. line AB be par[l] to the plane XY, and let AD, any plane through AB, cut the plane XY in the line CD.

It is required to prove that CD *is par[l] to* AB.

Proof. Now AB, being par[l] to the plane XY, can never meet CD which lies in that plane.

Moreover AB and CD are in the same plane AD;

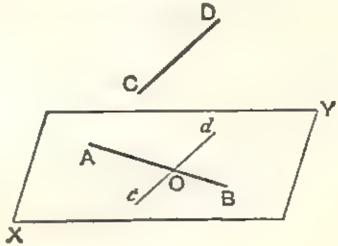
∴ AB and CD are par[l].

Q.E.D.

COROLLARY. *Through either of two skew straight lines a plane may be made to pass to which the other line is parallel.*

Let AB and CD be two skew st. lines, that is, two lines which are not co-planar.

Through any point O in AB let *cOd* be drawn par[l] to CD. Then AB and *cd* determine a plane XY, to which CD is par[l]; for CD is par[l] to the line *cd* which lies in that plane.



DEFINITION. The angle between two skew straight lines is measured by the angle contained by one of them and the line drawn through any point in that line parallel to the other.

Thus, in the above diagram, the angle between the skew lines AB, CD is measured by the angle contained by AB and the line *cd* drawn through any point O in AB parallel to CD.

EXERCISES.

1. If a straight line AB is parallel to a plane XY, then

    (i) *Every line parallel to* AB *is also parallel to the plane;*

    (ii) *Every line parallel to the plane is also parallel to* AB.

Which of these statements is true, and which false?

2. A straight line AB revolves about the point A, keeping always parallel to a given plane XY. What surface does AB generate?

3. If two intersecting planes pass respectively through two parallel lines AB, CD, shew that their line of section is parallel to AB, CD.

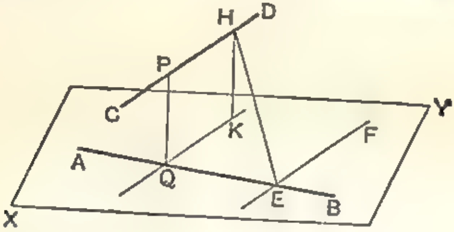
4. A straight line PQ is parallel to each of two intersecting planes; shew that PQ is also parallel to their line of section.

5. *Shew that through a given point* P *a plane may be constructed parallel to each of two skew lines* AB, BC.

6. *Shew that through any two given skew lines two parallel planes may be passed, one through each line.*

Theorem 94.

If two straight lines neither intersect nor are parallel, then
(i) *there is one straight line perpendicular to both of them;*
(ii) *this common perpendicular is the shortest distance between the given lines.*



Let AB and CD be the two given skew lines.

(i) *To prove that there is one line perp. both to* AB *and* CD.

Through E, any point in AB, let EF be drawn par' to CD, and let XY be the plane of AB, EF.

Suppose the projection of CD on the plane XY to be QK, cutting AB at Q: and let P be the point of which Q is the projection.

Then PQ will be perp. both to AB and CD.

Proof. Now CD, being par' to EF, is par' to the plane XY;
∴ CD is also par' to its projection QK. *Theor.* 93.
And because PQ is perp. to the plane XY,
∴ each of the ∠s PQB, PQK is a rt. angle;
∴ the ∠ QPD is a rt. angle; *Theor.* 13.
that is, PQ is perp. both to AB and CD.

(ii) *To prove that* PQ *is the shortest distance between* CD *and* AB.

Let HE be *any* other st. line drawn from CD to AB; and let HK be the perp. from H to the plane XY.

Then the perp. HK is less than the oblique HE. *Theor.* 85.

∴ PQ, which is equal and par' to HK, is also less than HE.

Q.E.D.

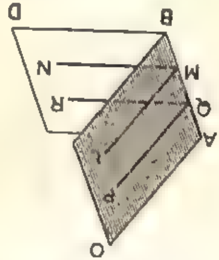
DEFINITIONS.

1. When two planes meet along a line of section they are said to form a dihedral angle.

2. A dihedral angle is measured by the plane angle contained by two straight lines drawn from *any* point in the line of section at right angles to it, one in each plane.

Thus in the adjoining Figure, AB is the line of section of the two intersecting planes BC, AD; and from Q, *any* point in AB, st. lines QP, QR are drawn perpendicular to AB, one in each plane.

Then the dihedral angle formed by the two planes is measured by the angle PQR.



NOTES. (i) This definition assumes that whatever point Q is taken in AB, the angle PQR is of constant magnitude: the truth of which follows readily from Theorem 87.

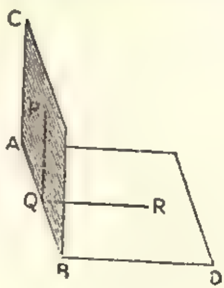
For let ML, MN be drawn perp. to AB, one in each plane, from any other point M in AB.

Then clearly, LM, MN are respectively par[l] to PQ, QR;

$\therefore$ the $\angle$LMN = the $\angle$PQR.

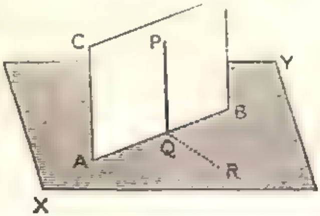
(ii) Since AB is perp. to PQ and QR, it is perp. to the plane containing them. Hence the dihedral angle between two given planes BC, AD may be determined by cutting them with *any* plane perp. to AB their line of section.

3. Two planes are perpendicular to one another when the dihedral angle formed by them is measured by a right angle.



Theorem 95. [Euclid XI. 18.]

If a straight line is perpendicular to a plane, then any plane passing through the perpendicular is also perpendicular to the given plane.



Let the st. line PQ be perp. to the plane XY, and let CB be any plane passing through PQ.

It is required to prove that the plane CB *is perpendicular to the plane* XY.

Proof. Let QR be drawn in the plane XY perp. to AB the line of section of the given planes.

Then PQ, being perp. to the plane XY, is perp. to QB and QR.

Thus the ∠PQR is a rt. angle; moreover the ∠PQR measures the dihedral angle, since PQ, QR are both perp. to the line of section AB.

∴ the plane CB is perp. to the plane XY.

Q.E.D.

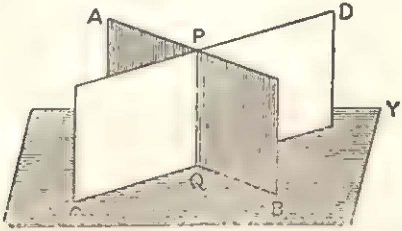
Corollaries. With the same construction, it may be shewn conversely that:

(i) *If the two planes* CB, XY *are perpendicular to one another, then any line* PQ *drawn in the plane* CB *perpendicular to the line of section* AB *will also be perpendicular to the plane* XY.

(ii) *If the plane* CB *is perpendicular to the plane* XY, *and from any point* P *in the first plane a perpendicular* PQ *is drawn to the second, then* PQ *is contained in the plane* CB.

Theorem 96. [Euclid XI. 19.]

If two intersecting planes are each perpendicular to a third plane, their line of section is also perpendicular to that plane.



Let each of the planes AB, CD, whose line of section is PQ, be perp. to the plane XY.

It is required to prove that PQ *is perp. to the plane* XY.

Proof. If from any point P, common to the planes AB, CD, a perpendicular is drawn to the plane XY, then this perpendicular must lie in each of the planes AB, CD, for each plane is perp. to XY. *Theor.* 95. *Cor.* (ii).

Hence the perpendicular must coincide with PQ the line of section; that is, PQ is perp. to the plane XY. Q.E.D.

EXERCISES.

1. Through any straight line a plane can be passed perpendicular to a given plane.

2. Prove that a straight line makes equal angles with any parallel planes which it cuts.

3. If a plane cuts two parallel planes, the corresponding dihedral angles are equal.

4. ABCD represents the floor of a room, and A′B′C′D′ its ceiling. If the length AB = 7·50 metres, the breadth AD = 6·00 metres, the height AA′ = 4·50 metres, find the cosine of the dihedral angles between

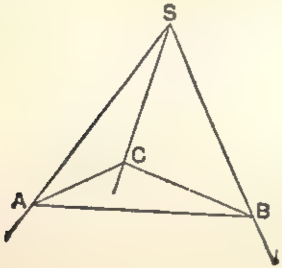
(i) the plane ABC′D′ and the floor;

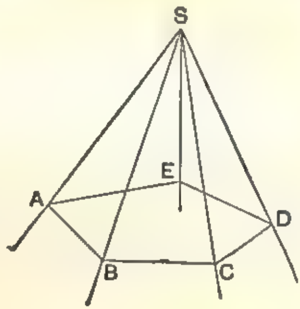
(ii) the plane AB′C′D and the floor.

5. P is a point 2 feet vertically above the in-centre of a horizontal square ABCD. If AB = 1 ft. 2 in., find the cosine of the dihedral angle between the plane PAB and the plane of the square.

SOLID ANGLES.

1. When three or more planes taken in order intersect each with the next, in lines which meet at a point, they are said to form a solid angle. The point of concurrence is called the **vertex**; the lines of intersection of consecutive planes are the **edges** of the solid angle; the angles between consecutive planes are its **dihedral angles**; and the plane angles formed by consecutive edges are its **face-angles.**



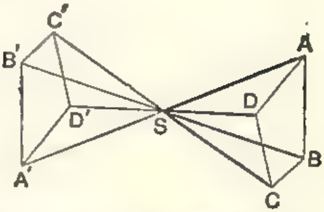


Thus the planes ASB, BSC, ..., cutting consecutively in the concurrent edges SB, SC, ..., form a solid angle at the vertex S. The solid angle is denoted by (S, ABCDE), or by the single letter S.

2. A solid angle formed by *three* concurrent planes is said to be **trihedral**; if formed by more than three, it is said to be **polyhedral.** The face angles and dihedral angles of a *trihedral* angle are known as its **six parts.**

3. Two solid angles are **identically equal** when one can be superposed upon the other, that is to say, exactly fitted into it; in which case the face-angles of the first solid angle are severally equal to the face-angles of the other, and the dihedral angles of the first to the dihedral angles of the other, these parts being taken in order *the same way round.*

NOTE 1. The necessity for the last condition may be seen on comparing a solid angle with that det rmined by producing its edges thr ugh the vertex.



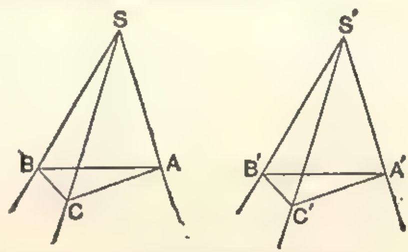
Here the solid angles (S, ABCD), (S, A′B′C′D′) have the face-angles and dihedral angles of one severally equal to the face-angles and dihedral angles of the other in the order indicated by the letters. But to an observer looking *into* each solid angle in turn the above sequence of letters will appear *clock-wise* in the first case, and *counter-clockwise* in the second. Hence the two solid angles, though equal as to their several parts, cannot be fitted one into the other, and are therefore not identically equal. Solid angles so related are said to be symmetrical.

NOTE 2. *If two* trihedral *angles* (S, ABC), (S′, A′B′C′) *have the three face-angles* ASB, BSC, CSA *in one equal respectively to the three face-angles* A′S′B′, B′S′C′, C′S′A′ *in the other, the dihedral angles of the first will be equal to the corresponding dihedral angles of the other.*

Make SA, S′A′ of any equal lengths.

In the planes ASB, ASC draw AB, AC each perp. to SA.

In the planes A′S′B′, A′S′C′ draw A′B′, A′C′ each perp. to S′A′.



Then the $\angle$s BAC, B′A′C′ measure corresponding dihedral angles.

Join BC, B′C′

Outline of Proof. Prove the following pairs of triangles congruent :

(i) the $\triangle$s SAB, S′A′B′; (*Theor.* 17.)

(ii) the $\triangle$s SAC, S′A′C′; (*Theor.* 17.)

(iii) the $\triangle$s BSC, B′S′C′; (*Theor.* 4.)

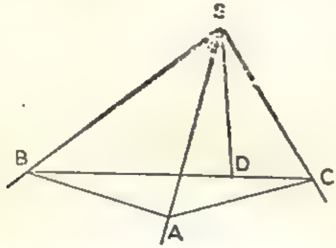
(iv) the $\triangle$s BAC, B′A′C′; (*Theor.* 7.)

hence the $\angle$ BAC = the $\angle$ B′A′C′.

Similarly the remaining dihedral angles may be proved equal; and if the sequence of equal angles goes the same way round in each case, the two trihedral angles are identically equal.

THEOREM 97. [Euclid XI. 20.]

In a trihedral angle the sum of any two of the face-angles is greater than the third.



Let (S, ABC) be a trihedral angle, of which ASB, BSC, CSA are the face-angles; and of these let the ∠ BSC be the greatest.

It is enough to prove that

the sum of the ∠ˢ ASB, ASC *is greater than the* ∠ BSC.

In the plane BSC make the ∠ BSD equal to the ∠ BSA; and cut off SD, SA of any equal lengths.

In the plane BSC draw any st. line through D cutting SB, SC at B and C; and join AB, AC.

Proof. Then in the △ˢ BSA, BSD,

since BS, SA are equal to BS, SD respectively,

and the ∠ BSA = the ∠ BSD,

∴ BA = BD. *Theor.* 4.

Now from the △ BAC,

BA + AC is greater than BC,

that is, greater than BD + DC;

∴ AC is greater than DC.

Again, in the △ˢ ASC, DSC,

since AS, SC = DS, SC respectively,

but AC is greater than DC,

∴ the ∠ ASC is greater than the ∠ DSC. *Theor.* 19. *Cor.*

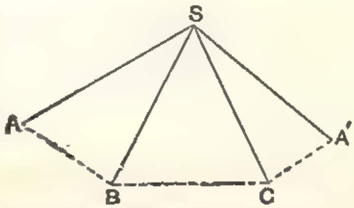
∴ the sum of the ∠ˢ ASB, ASC is greater than the sum of the ∠ˢ BSD, DSC,

that is, greater than the ∠ BSC.

Q.E.D.

EXPERIMENTAL ILLUSTRATION OF THEOREM 93.

To construct a solid angle, draw the three face-angles ASB, BSC, CSA′ *in the same plane*, placing the greatest angle BSC between the other two.



Suppose the diagram cut out, and folded about SB and SC, with a view to bringing SA and SA′ into coincidence.

Now (i) if $\angle BSA + \angle CSA'$ is less than $\angle BSC$,
then SA and SA′ cannot be brought together, and therefore no solid angle can be formed;

(ii) if $\angle BSA + \angle CSA' = \angle BSC$,
then SA and SA′ can be made to coincide, but *in the plane of the* $\angle BSC$, so that no solid angle can be formed;

(iii) if $\angle BSA + \angle CSA'$ is greater than $\angle BSC$,
then SA and SA′ would overlap when brought into the plane BSC, so that they can be made to coincide outside that plane, and a solid angle can consequently be formed.

EXERCISES.

1. Prove that in general three planes meet at a point. Indicate three exceptional cases.

2. Shew that the sum of the angles of a *skew* quadrilateral is less than 360°.

3. OA, OB, OC are three straight lines drawn from a given point O not in the same plane, and OX is another straight line within the solid angle determined by OA, OB, OC: shew that

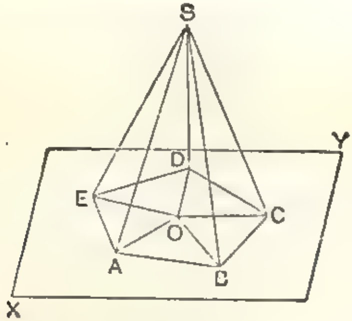
(i) the sum of the angles AOX, BOX, COX is greater than half the sum of the angles AOB, BOC, COA.

(ii) the sum of the angles AOX, COX is less than the sum of the angles, AOB, COB.

(iii) the sum of the angles AOX, BOX, COX is less than the sum of the angles AOB, BOC, COA.

THEOREM 98. [Euclid XI. 21.]

In a convex solid angle the sum of the face angles is less than four right angles.



Let (S, ABCDE) be a convex solid angle.

It is required to prove that the sum of the face-angles ASB, BSC, CSD, DSE, ESA *is less than four right angles.*

Let a plane XY cut the planes of the face-angles in the lines AB, BC, CD, DE, EA, which form the convex polygon ABCDE.

Within the polygon ABCDE take any point O, and join OA, OB, OC, OD, OE.

Proof. In the trihedral angle A,

$\angle$SAB + $\angle$SAE is greater than $\angle$EAB, *Theor.* 97.

that is, greater than $\angle$OAE + $\angle$OAB.

Similarly for each of the angular points, B, C, D, E.

Hence the sum of the base $\angle$s of the $\triangle$s with vertex S is greater than the sum of the base $\angle$s of the $\triangle$s with vertex O.

And since these two sets of $\triangle$s are equal in number, the sum of all the $\angle$s of one set is equal to the sum of all the $\angle$s of the other.

It follows that the sum of the angles at S *is less than* the sum of the angles at O.

But the sum of the angles at O is 4 rt. $\angle$s;

$\therefore$ the sum of the angles at S is less than 4 rt. $\angle$s.

EXERCISES.

(*Miscellaneous.*)

1. *Shew that the angle made by an oblique to a plane with its projection is less than that which it makes with any other line which meets it in that plane.*

2. *Through any point on an inclined plane shew how to draw on that plane the* LINE OF GREATEST SLOPE (that is, the line making the greatest angle with the horizontal plane).

3. O is a fixed point in a plane, and P a fixed point outside it. Find the locus of the feet of perpendiculars drawn from P to all lines in the plane through O.

4. From a point A two normals AP, AQ are drawn one to each of two intersecting planes: shew that

(i) the line of section is perpendicular to the plane of AP, AQ;

(ii) the dihedral angle between the two planes is equal or supplementary to the angle between the normals.

5. If AB and CD are two skew lines, shew that the joining lines AC, BD are also skew.

On what planes will the projections of the skew lines AB, CD be parallel?

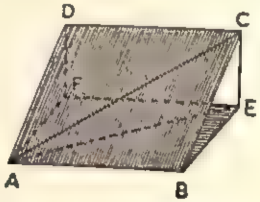
6. Shew that through any point in space there is always one straight line which cuts each of two given skew lines.

7. OA, OB, OC are three concurrent lines, each of which is perpendicular to the other two; then

(i) if OX, OY, OZ are perpendicular to BC, CA, AB respectively, shew that XYZ is the pedal triangle of the triangle ABC;

(ii) if OP is perpendicular to the plane of ABC, shew that P is the orthocentre of the triangle ABC.

8. ABCD is an inclined plane; AB, CD horizontal lines in it; AD, BC lines of greatest slope in the plane; AF, BE the projections of AD, BC on the horizontal plane.

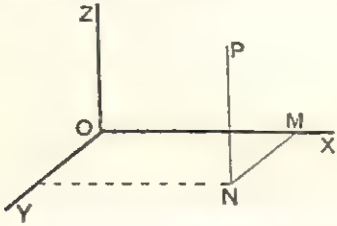


If $\angle CBE=\alpha$, $\angle DAC=\beta$, $\angle CAE=\theta$, $\angle FAE=\phi$, prove the following relations:

(i) $\sin\theta=\sin\alpha\cos\beta$; (ii) $\tan\phi=\tan\beta\sec\alpha$;

(iii) $\tan\alpha=\tan\theta\sec\phi$; (iv) $\sin\beta=\sin\phi\cos\theta$.

NOTE ON FIXING THE POSITION OF A POINT IN SPACE BY MEANS OF AXES OF REFERENCE.



Let OX, OY be two fixed st. lines cutting at right angles at the origin O. Draw OZ perp. to the plane of OX, OY. Then each of the lines OX, OY, OZ is perp. to the other two, and consequently to the plane determined by the other two. The lines OX, OY, OZ are taken as axes of reference, and the position of any point P with regard to them is fixed in the following manner.

Let N be the projection of P on the plane XOY; let OM, MN be the coordinates of N with reference to OX, OY; and let x, y, z denote the lengths of OM, MN, NP. Then OM, MN, NP, taken together, are said to be the coordinates of P; the point P is denoted by (x, y, z), and its position is known if the numerical values of x, y, and z are given.

Example. *Plot the point whose coordinates are* 5, 3, and 4.

First, with reference to the axes OX, OY, plot the point whose coordinates are 5, 3: call this point N, and draw NP perp. to the plane XOY, making NP 4 units of length. This gives the position of P.

It will be seen that the coordinates of a point P are *its distances from the three planes of reference* YOZ, ZOX, XOY. Now these planes divide space into *eight* regions, and in each of these there is a point whose distances from the planes are 5, 3, and 4. The coordinates of these points are distinguished by the use of signs on principles analogous to those explained on page 133.

Lines measured along or parallel to the axes OX, OY, OZ, in the *senses* indicated by the letters, are *positive*. Lines measured in these directions but in senses opposite to the above are negative.

EXERCISES.

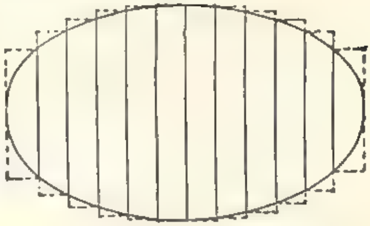
1. Draw diagrams to indicate the positions of the following points:
(i) (3, 5, 4); (ii) (−5, 4, 3); (iii) (−3, −4, 5); (iv) (4, 5, −3).

2. The coordinates of P are 6, 8, 10; find the coordinates of Q, the mid-point of OP.

3. If P is the point (x, y, z), shew that $OP^2 = x^2 + y^2 + z^2$.
Find OP when P is given by (3, 4, 12).

SOLID FIGURES.

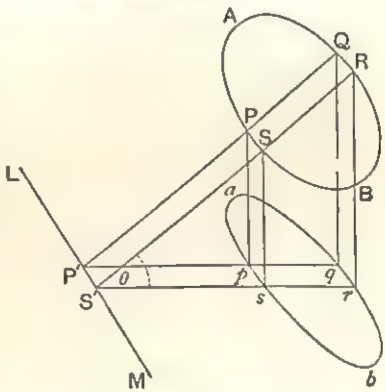
Definitions and Preliminary Theorems.



1. If a plane figure is cut by a system of equidistant parallel lines, and if the rectangle is completed between each pair of consecutive parallels, as shewn in the diagram, then the area of the external rectilineal figure may be made to differ from the area of the given figure by as little as we please by indefinitely diminishing the width of the strips; that is, *the area of the given figure may be regarded as the limit of the sum of all such rectangular strips, when their breadth is diminished indefinitely.*

2. If AB is any plane figure, and ab its projection on a second plane making an angle θ with the first, then

area of the projection $ab =$ (*area of the given fig.* AB) $\times \cos\theta$.



For the fig. AB may be divided into narrow strips by parallels drawn perpendicular to the section LM of the planes. Let PQRS be any such strip, and $pqrs$ its projection.

Then the strips $pqrs$, PQRS, when taken very small may be regarded as rectangular and as having the same breadth (viz. P′S′), and pq the length of the former, is PQ $\cos\theta$;

$\therefore$ area of strip $pqrs =$ (area of PQRS) $\times \cos\theta$.

And this is true for every pair of corresponding strips in the limit when their breadth is diminished indefinitely;

$\therefore$ *area of fig.* $ab =$ (*area of fig.* AB) $\times \cos\theta$.

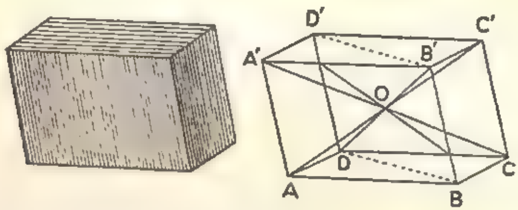
3. A **solid figure**, or **solid**, is any portion of space bounded by one or more surfaces, plane or curved.

These surfaces are called the **faces** of the solid, and the intersections of adjacent faces are called **edges**.

4. A **polyhedron** is a solid bounded by plane faces.

NOTE. A plane rectilineal figure must at least have *three* sides; or *four*, if two of the sides are parallel. A polyhedron must at least have *four* faces; or, if two faces are parallel, it must at least have *five* faces.

5. A **parallelepiped** is a solid bounded by three pairs of parallel plane faces.



Since in Fig. 2 the parallel planes ABCD, A′B′C′D′ are cut by the plane ABA′B′, the edges AB, A′B′ are parallel [*Theor.* 89].

Thus it may be shewn (i) that each of the six faces of a parallelepiped is a parallelogram; (ii) that opposite faces are congruent; (iii) that the twelve edges fall into three groups, each group of four edges being equal and parallel to one another.

6. *The four diagonals of a parallelepiped are concurrent and bisect one another.*

Let AC′, BD′, CA′, DB′ be the diagonals of the parallelepiped (ABCD, A′B′C′D′).

Join BD, B′D′.

Then since BB′, DD′ are equal and par[l], the fig. BDD′B′ is a par[m];

∴ its diagonals BD′, DB′ bisect one another:

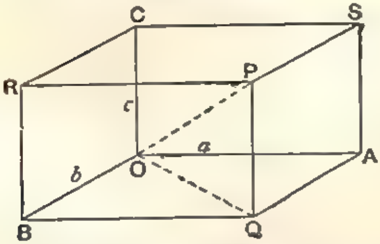
that is, BD′ passes through O the middle point of DB′.

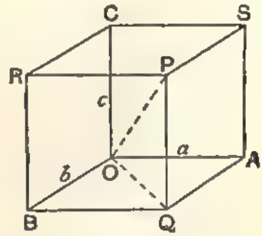
Similarly, by joining DA′, B′C, it may be shewn that A′C passes through O the middle point of DB′.

In the same way AC′ is also bisected at O.

7. A parallelepiped whose faces are rectangular is called a **cuboid**, or **rectangular solid**.

If the faces are all squares the parallelepiped becomes a **cube**





Since in the above Figures the ∠s COA, COB are rt. ∠s,

∴ OC is perp. to the face AB.

Similarly each edge is perp. to the two faces which it cuts;
and each face is perp. to the four faces which it cuts.

8. *The square on a diagonal of a rectangular solid is equal to the sum of the squares on three concurrent edges.*

Let OP be a diagonal of a cuboid, in which three concurrent edges OA, OB, OC measure, a, b, and c units of length. Join OQ.

Then PQ, being perp. to the face AB, is perp. to OQ.

$$\therefore\ OP^2 = OQ^2 + PQ^2 = OQ^2 + c^2.$$

But $OQ^2 = OA^2 + AQ^2 = a^2 + b^2$, for OAQ is a rt. ∠.

$$\therefore\ OP^2 = a^2 + b^2 + c^2.$$

COROLLARY 1. *The diagonals of a cuboid are equal.*

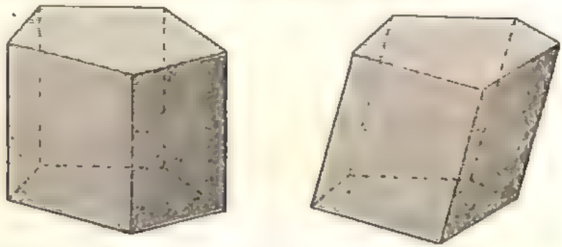
COROLLARY 2. *If* a *denotes each edge of a cube; then*

$$(diagonal)^2 = 3a^2; \qquad \therefore\ diagonal = a\sqrt{3}.$$

NOTE. If (x, y, z) are the coordinates of a point P, then, O being the origin,

$$OP^2 = x^2 + y^2 + z^2.$$

9. A **prism** is a solid bounded by plane faces, of which two called the **ends**, are congruent figures in parallel planes; and the others, called **side-faces** are parallelograms.



The ends of a prism may be triangles, quadrilaterals, or polygons of any number of sides; and the corresponding prisms are said to be *triangular*, *quadrilateral*, or *polygonal.*

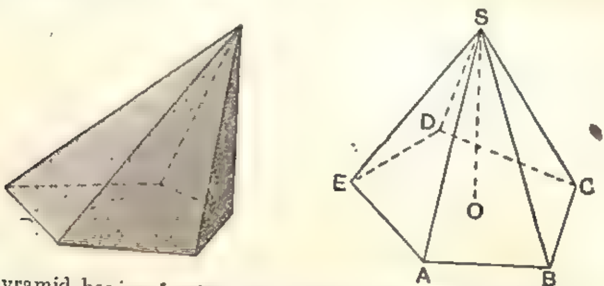
The side-edges of every prism are all parallel and equal.

A prism is said to be **right**, if the side-edges are perpendicular to the ends: In this case the side-faces are rectangles. All other prisms are said to be **oblique.**

A parallelepiped is a particular form of prism. Cuboids and cubes are special forms of *right* prisms.

It follows from Theorem 89 that a plane section of a prism parallel to either end is identically equal to each end.

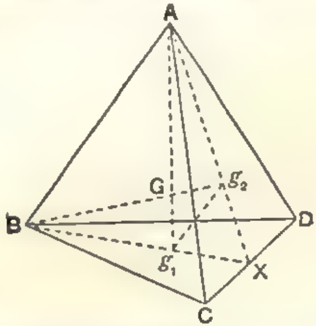
10. A **pyramid** is a solid bounded by plane faces, of which one, called the **base**, is any rectilineal figure, and the rest are triangles having a common vertex at some point not in the plane of the base.



A pyramid having for its base a *regular* polygon ABCDE is said to be right when the vertex S lies in the straight line SO drawn perpendicular to the base from its central point O (the centre of its inscribed or circumscribed circle).

11. **A tetrahedron** is a pyramid on a triangular base: it is thus contained by *four* triangular faces.

12. *The four lines which join the vertices of a tetrahedron to the centroids of the opposite faces meet at a point which divides them in the ratio* 3 : 1.



In the tetrahedron (A, BCD), let g_1, g_2, g_3, g_4 be the centroids of the faces opposite respectively to A, B, C, D.

To prove that Ag_1, Bg_2, Cg_3, Dg_4 *are concurrent.*

Take X the middle point of the edge CD;
then g_1 and g_2 must lie respectively in BX and AX,
so that $BX = 3Xg_1$, and $AX = 3Xg_2$; [Page 97]
$\therefore$ g_1g_2 is par' to AB.

And Ag_1, Bg_2 must intersect one another, since they are both in the plane of the $\triangle$ AXB:
let them intersect at the point G.

Then by similar $\triangle$', $\quad AG : Gg_1 = AB : g_1g_2$
$= AX : Xg_2$
$= 3 : 1.$

$\therefore$ Bg_2 cuts Ag_1 at a point G whose distance from $g_1 = \frac{1}{4} . Ag_1$.

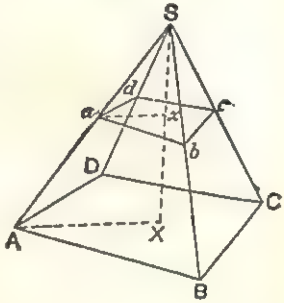
Similarly it may be shewn that Cg_3 and Dg_4 cut Ag_1 at the same point; $\therefore$ these lines are concurrent.

EXERCISE. *The three lines which join the middle points of opposite edges of a tetrahedron are concurrent and bisect one another.*

[Let X, Y, Z, V be the mid-points of CD, DA, AB, BC.
Prove the fig. XYZV to be a par^{m} (see Ex. 2, p. 362), and consider its diagonals].

13. (i) *Any plane section of a pyramid taken parallel to the base is similar to the base.*

(ii) *The area of such a section varies as the square of its distance from the vertex.*



In the pyramid (S, ABCD), let $abcd$ be a plane section par[l] to the base ABCD.

(i) Because the plane abAB meets the parallel planes $abcd$, ABCD, the lines of section ab, AB are parallel.

Similarly bc and BC, cd and CD, da and DA are parallel;

∴ corresponding angles of the figs. $abcd$, ABCD are equal.

And, from similar △^{s}, $\quad \dfrac{ab}{AB} = \dfrac{bc}{BC} = \dfrac{cd}{CD} = \dfrac{da}{DA}$;

hence the figs. $abcd$, ABCD are similar.

(ii) Let the perp. drawn from S to the base, meet the sections $abcd$, ABCD at x and X. Join ax, AX.

Then fig. $abcd$: fig. ABCD $= ab^2 : AB^2$ *Theor.* 73.

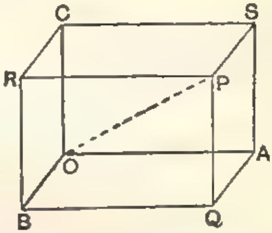
$= Sa^2 : SA^2$, by similar △^{s},

$= Sx^2 : SX^2$, ...

Corollary. *If two pyramids stand on bases of equal area and have equal altitudes, then plane sections taken in each pyramid parallel to the base and at the same distance from the vertex are equal in area.*

EXERCISES.

1. A square sheet of galvanized iron, each side being 12 feet, rests against a wall, and is inclined to the horizon at an angle of 60°; what area of ground will it protect from a vertical rain?



2. (i) In the rectangular solid represented in the margin, if OA = 12 cm., OB = 9 cm., OC = 8 cm., find the values of OP, cos QOP, area of fig. OAPR.

(ii) If OP makes with OA, OB, OC the angles α, β, γ, respectively, shew that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,;$$

and verify this result with the dimensions given above.

(iii) What plane passing through OP is parallel to BQ? If OA $= a$, OB $= b$, OC $= c$, shew that the shortest distance between OP and BQ is $bc/\sqrt{b^2+c^2}$.

3. If a parallelopiped is cut by a plane which intersects two pairs of opposite faces, shew that the lines of section form a parallelogram.

4. Shew that the polygons formed by cutting a prism by parallel planes are identically equal.

5. If each edge of a tetrahedron is equal to the opposite edge, shew that the sum of the plane angles in each corner is 180°.

6. Two planes cut at an angle of 45°. On one plane a circle of radius 5 cm. is drawn, and this circle is projected on the other plane.

Find (i) the length of the greatest chord, (ii) the area of the projection.

7. If a tetrahedron is cut by any plane parallel to two opposite edges, the section will be a parallelogram.

8. Prove that the shortest distance between two opposite edges of a regular tetrahedron is one half of the diagonal of the square on an edge.

9. In a tetrahedron if two pairs of opposite edges are at right angles, then the third pair will also be at right angles.

10. In a tetrahedron whose opposite edges are at right angles, the sum of the squares on each pair of opposite edges is the same.

SURFACES AND VOLUMES.

14. The **volume** of a solid is the amount of space contained within its bounding surfaces.

A **cubic inch** is the volume of a cube each of whose edges is one inch in length. Similarly a cubic centimetre is the volume of a cube of which each edge is one centimetre in length. Thus the unit of volume is the volume of a cube on an edge of unit length.

15. *To find the* surface *and* volume *of a* rectangular solid.

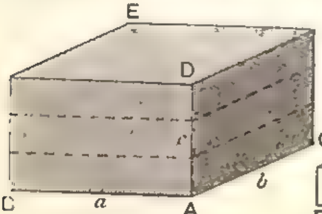


Fig. 1.

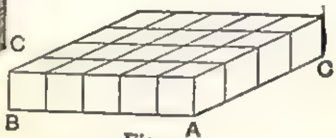


Fig. 2.

Surface. In the cuboid represented in Fig. 1, let the length AB $= a$ units, the breadth AC $= b$ units, and the height AD $= c$ units.

Then the whole surface is the sum of the three pairs of opposite and equal rectangular faces.

Now the faces DE, DB, DC contain respectively ab, ac, bc, units of area;

$\therefore$ *whole surface of cuboid* $= 2ab + 2ac + 2bc$ units of area.

If $a = b = c$, the rectangular solid becomes a cube on an edge of a units, and *whole surface of cube* $= 6a^2$ units of area.

Volume. Consider a cuboid whose length AB is 5 inches, breadth AC 4 inches, height AD 3 inches. Fig. 1 shews that the solid may be divided into 3 equal *slices*, each 1 inch thick. And each slice may be subdivided (as in Fig. 2) into cubical blocks whose edges are 1 inch, that is, into cubic inches.

Now the number of cubic inches in one slice is 5×4; so that the number of cubic inches in the whole solid is $5 \times 4 \times 3$, or 60.

Similarly, if the length $= a$ linear units, the breadth $= b$ linear units, and the height $= c$ linear units,

the rectangular solid contains **abc** *units of* **volume.**

And if each edge of a cube $= a$ linear units,

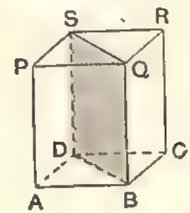
the tube contains **a³** *units of* **volume.**

These statements may be thus abridged :

volume of cuboid = length × breadth × height ....(i)

= (area of base) × height. ....(ii)

volume of cube = (edge)³. ....(iii)



COROLLARY. The cuboid (ABCD, PQRS) is divided by the diagonal plane BDSQ into two right prisms whose bases are congruent right angled triangles. These prisms are identically equal, and the volume of each is half that of the complete cuboid.

EXERCISES.

1. The length, breadth, and height of a room being respectively a, b, and c units, shew that the four walls contain $2c(a+b)$ units of area.

If the area of the four walls is equal to 86·70 sq. metres, and the height is 3·40 metres, find the perimeter of the floor.

2. Express in litres the capacity of a rectangular cistern whose length, breadth, and depth are respectively 125 cm., 80 cm., and 65 cm.

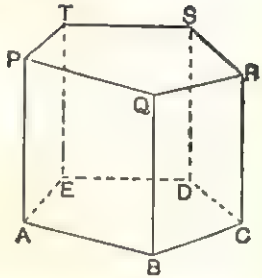
Find in kilograms the weight of water required to fill the cistern to four-fifths of its capacity.

3. The annual rainfall at a certain place is 65 cm.; to how many litres per hectare is this equivalent ?

4. What is the weight of a rectangular block of granite whose dimensions are 1·20 metres, 0·75 metres, 0·50 metres, at the rate of 2·64 kg. per cubic decimetre ?

[For further Exercises on Rectangular Solids see p. 393.]

16. *To find the lateral* **surface** *of a* **right prism.**



In the given prism let the sides of the base AB, BC, CD, ... contain a, b, c, ... units of length, and let the height $=h$. Then, the prism being *right*, each side-edge $=h$, and each side-face is a rectangle.

Then the area of the rect. ABQP $=ah$, and the areas of the remaining side-faces are bh, ch, ...

$\therefore$ *lateral surface of prism* $= ah + bh + ch + \ldots$

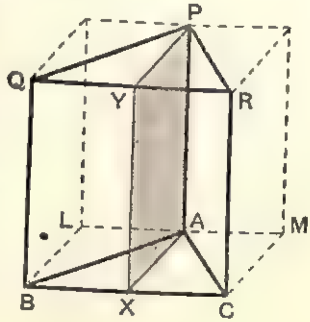
$= (a+b+c+\ldots)h$ units of area

$=$ (*perimeter of base*) $\times$ *height.*

17. *To find the* **volume** *of a* **right prism.**

(i) First consider a prism (ABC, PQR) on a *triangular* base ABC; let its height be h.

Through AP draw the plane APYX perp. to the face BCRQ, thus dividing the given prism into two prisms whose bases are the right-angled $\triangle^s$ AXB, AXC. Through A draw LM par[l] to BC; complete the rect. BLMC, and on BLMC as base construct a cuboid of height h.



Then prism on base AXB $= \frac{1}{2}$ (cuboid on base AXBL);

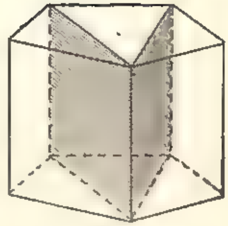
and prism on base AXC $= \frac{1}{2}$ (cuboid on base AXCM);

$\therefore$ *given prism on base* ABC $= \frac{1}{2}$ (cuboid on base LBCM)

$= \frac{1}{2}$ rect. LBCM $\times$ height

$=$ (*area of base* ABC) $\times$ *height.*

(ii) Now a prism on a polygonal base may always be divided, as shewn in the diagram, into a set of prisms on triangular bases and having the same height as the given prism;



$\therefore$ *volume of* any *right prism* = (sum of triangular bases) × height.
= *(base of given prism)* × *height.*

EXERCISES ON RECTANGULAR SOLIDS AND RIGHT PRISMS.

[1 *litre is equivalent to* 1 *cubic decimetre.*

1 *cubic decimetre of water weighs* 1 *kilogram.*

The specific gravity of a substance is the ratio of the weights of equal volumes of the given substance and water.

Thus if the specific gravity of steel is 7·8, it follows that the weight of 1 cubic decimetre of steel is 7·8 kilograms.]

1. The area of the section of a boring is 1325 square feet, and the excavating machine is driven forward 4 feet a day. How many cubic yards of earth are excavated in a day?

2. Find in kilograms the weight of water in a trench, whose length and breadth are 21·25 metres and 1·50 metres, the depth of the water being 64 cm.

3. Find the weight of a steel bar 1·28 metre long, 15 cm. wide, and 5 cm. thick, the specific gravity of steel being 7·8.

4. A level seam of coal has an average thickness of $3\frac{1}{2}$ feet: find approximately the yield in tons per acre.

[1 cu. ft. of water weighs 1000 oz., and the specific gravity of coal = 1·28.]

EXERCISES. (*Continued.*)

(*On Rectangular Solids.*)

5. Find to the nearest penny the cost of painting the four inner sides and bottom of a tank, which measures internally 2·50 metres long, 1·24 metres wide, and 1·50 metres deep, at 7*d.* per square metre.

6. Find the weight per square metre of sheet zinc 3 mm. thick, the specific gravity of zinc being 7·14.

7. A chest whose external length, breadth, and height are respectively 1·65 metres, 1·25 metres, and 0·55 metres, is made of oak 25 mm. thick. What are its inner dimensions? Find to the nearest penny the cost of lining the sides and bottom with thin metal at 1s. 3d. per square metre.

8. What is the length of the edge of a cube of which

(i) the surface is 2·5350 square metres?

(ii) the volume is 274,625 cubic cm.?

9. A closed box is built of wood of uniform thickness. Its external dimensions are 12 cm., 10 cm., and 8 cm.; and the *inner* surface is 376 sq. cm.; find the thickness of the wood.

10. The whole surface of a rectangular block is 1332 sq. cm. Find the length, breadth, and height; it being given that these dimensions are proportional to the numbers 4, 5, 6.

11. The whole surface of a rectangular block is 214 sq. cm. The base contains 42 sq. cm., and one vertical face contains 35 sq. cm. Find the edges.

12. Find to the nearest millimetre the edge of a cube whose diagonal is 10 cm. Find also the whole surface and volume of the cube.

13. A rectangular block stands on a base of 48 sq. cm.; its height is 3 cm., and its diagonal is 13 cm. Find the length and breadth.

14. The diagonal of a rectangular solid is 17 cm. and the whole surface is 552 sq. cm. Find the sum of the three dimensions.

15. A rectangular tank stands on a base measuring 20 feet by 16 feet. If water flows in from a pipe which admits 40 gallons a minute, find at what rate (in inches per hour) the water will rise, reckoning $6\frac{1}{4}$ gallons as a rough equivalent of 1 cubic foot.

(On Right Prisms.)

16. The base of a right prism is a triangle ABC right-angled at C. If AC = 15 cm., CB = 8 cm., and the height of the prism = 12 cm., find the volume and lateral surface.

17. A right prism stands on a triangular base whose sides are 17 cm., 10 cm., 9 cm.; and the height is 10 cm. Find the volume and whole surface.

18. The base of a right prism is a trapezium whose parallel sides are 17 cm. and 13 cm., the distance between them being 8 cm. If the height of the prism is 1 metre, find the volume in cubic centimetres.

19. Sand lies against a wall covering a strip of ground 4 feet wide, and resting with its surface inclined at 30° to the horizon. Find, to the nearest tenth of a cubic foot, how much sand may lie on this strip per foot length of wall.

20. The vertical cross-section of a trench is a trapezium, measuring 15 feet across the top, 9 feet across the bottom; depth of trench uniform 8 feet; length of trench $62\frac{1}{2}$ feet. Find roughly how many gallons and how many tons of water the trench will hold.

[1 cubic foot of water is equivalent to $6\frac{1}{4}$ gallons nearly, and weighs a little less than 1000 ounces.]

21. A bed of coal 14 feet thick is inclined at 23° to the surface. Calculate the number of tons of coal that lie under an acre of surface. [The thickness is to be measured perpendicular to the coal-bed. One ton of coal occupies 28 cubic feet. Cos 23° = 0·9205.]

22. Through a wooden pipe, whose cross-section is a square on a side of 8 cm., water flows uniformly at the rate of 40 metres a minute. How long will it take to discharge a million litres?

23. Compare the lateral surfaces and the volumes of the following right prisms:

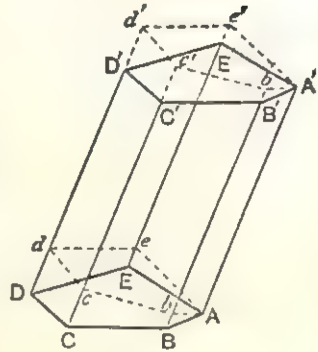
(i) base a regular hexagon on side 8 cm., height 6 cm.

(ii) base a regular octagon on side 6 cm., height 8 cm.

24. A railway cutting 850 metres in length is to have a uniform depth of 4·50 metres, and its dimensions across the top and bottom are to be respectively 31·20 metres and 16·80 metres. If 450 tons are excavated on an average per day, and 1 cubic metre weighs $2\frac{1}{4}$ tons, how long will the work require?

18. *To find the* **volume** *of an* **oblique prism**.

The diagram represents an oblique prism (ABCDE, A′B′C′D′E′), and *Abcde* is a *right* plane section, that is, a section perp. to all the side-edges.



Now suppose the slice cut off between the base ABCDE and the right section *Abcde* to be removed and placed on the other end A′B′C′D′E′, so that A falls on A′, B on B′, and so on; then the given oblique prism is converted into the *right* prism (*Abcde*, A′*b′c′d′e′*), whose edges are equal to those of the given prism, and whose volume = (area of its end *Abcde*) × AA′.

Hence *the volume of an oblique prism*

$$= (\textit{area of right section}) \times \textit{edge}. \qquad \text{(i)}$$

Now let θ be the angle between the base ABCDE and the right section *Abcde* (which is a projection of the base); then θ is also the angle between the perp. height h and the edge AA′, for the two lines are respectively normal to the two planes. Thus we have

right section *Abcde* = base ABCDE × cos θ; (Art. 2.)

also $h = AA' \cos \theta$.

Applying these values to (i),

Volume of oblique prism = base ABCDE × cos θ × AA′

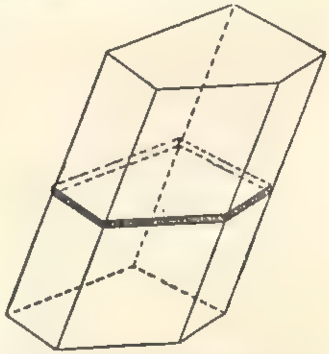
= base ABCDE × h.

Hence of *oblique* as of *right* prisms,

the volume = (*area of base*) × (*perpendicular height*).

We leave as an exercise to the student to prove that
the lateral surface of an oblique prism = (*perimeter of right section*) × *edge*.

19. Volume of oblique prism. [Alternative method.]



Cut the prism at equal intervals by a series of planes parallel to the base; and between each pair of consecutive planes, and on the section made by the lower of them, construct a *right* prism. Then the volume of such a prismatic slice = (area of its base) × thickness.

Now let the number of slices be increased indefinitely, and consequently the thickness of each one *diminished* indefinitely; then in the limit the whole prism is equivalent to the sum of all the slices.

But the base of each slice = the base of the prism; and the sum of the thicknesses = the perp. height.

Hence

volume of prism = (*area of base*) × (*perpendicular height*).

Corollary. *Prisms of the same perpendicular height and standing on bases of equal area are equal in volume.*

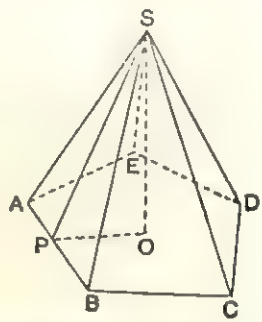
Note.—The above proof applies also to parallelepipeds, which are a special form of prism; the essence of the proof being the fact that *plane sections parallel to the base are identically equal.*

PYRAMIDS.

20. The **slant surface** of a pyramid (S, ABCDE) is the sum of the triangular faces SAB, SBC, SCD, ..., of which the areas in the general case must be separately found.

A convenient expression may however be obtained for the slant surface when the pyramid is *right* and on a *regular* base.

21. *To find the* **Slant surface** *of a* **right pyramid** *on a regular base of* n *sides.*



The pyramid being *right*, and the base *regular*, it follows that the slant edges SA, SB, SC, ... are all equal; and the faces SAB, SBC, SCD, ... are equal isosceles triangles.

SP, drawn perp. to a side of the base and therefore bisecting it, is called the *slant height* of the pyramid, and is the same for each slant face.

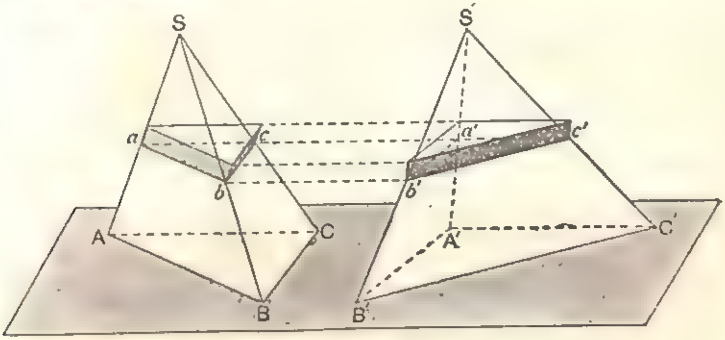
If SO is normal to the base, it follows (as in the Corollary to Theorem 83) that OP is perp. to AB.

Let each side of the base $= a$, the perp. height $\mathrm{SO} = h$, and the slant height $\mathrm{SP} = l$. Then

$$\begin{aligned} \textit{slant surface of pyramid} &= \triangle\,\mathrm{SAB} \times n \\ &= \tfrac{1}{2}\mathrm{AB}\,.\,\mathrm{SP} \times n \\ &= \tfrac{1}{2}\,.\,na \times l \text{ units of area} \\ &= \tfrac{1}{2}\,(\textit{perimeter of base}) \times (\textit{slant-height}). \end{aligned}$$

The *whole* surface = the slant surface + the area of the base.

22. *To prove that two pyramids* (S, ABC), (S′, A′B′C′) *standing on bases of equal area, and having equal perpendicular heights, are equal in volume.*



Place the pyramids with their bases ABC, A′B′C′ in the same plane, and cut them at equal intervals by a series of planes parallel to the plane of the bases.

Between each pair of consecutive planes, and on the section made by the lower of them, construct prisms with lateral edges parallel to SA and S′A′ respectively.

Then any pair of sections, such as *abc*, *a′b′c′*, made by the same plane, are equal in area (Art. 13. Cor.).

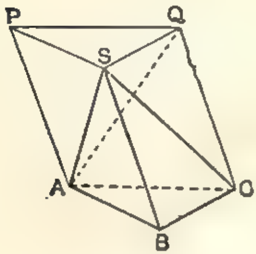
Hence the prismatic slices standing on these sections are of equal volume, for they are of the same thickness.

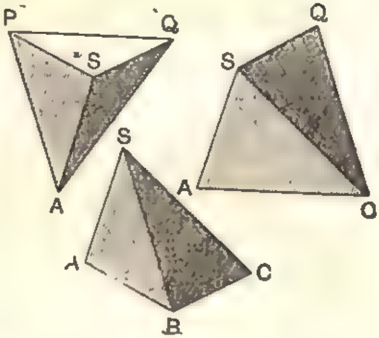
Now let the number of parallel sections be increased indefinitely, and consequently the thickness of the slices *diminished* indefinitely; then in the limit each pyramid is equivalent to the sum of its prismatic slices; and each slice in one is equal to the corresponding slice in the other.

Hence the whole pyramids are equal in volume.

Note. For the sake of simplicity triangular pyramids have been considered, but the reasoning is general.

23. *To find the* **volume** *of a triangular* **pyramid.**





Let (S, ABC) be a triangular pyramid of perp^r height h.

Through A and C draw par^ls to BS, and cut them by a plane through S par^l to the base ABC. This determines a triangular prism whose volume $= \triangle \text{ABC} \times h$.

Draw the diag. AQ of the par^m ACQP.

Now the prism may be broken up into the pyr^d (S, ABC), standing on the triangle-base ABC, and the pyr^d (S, ACQP), on the par^m-base ACQP.

But the latter may be broken up into the two pyr^ds (S, APQ), (S, ACQ), which have equal volumes since they stand on equal bases, and have the same vertex.

Again, regarding the pyr^d (S, APQ) as (A, PSQ), we see that

$$\text{pyr}^d(\text{A, PSQ}) = \text{pyr}^d(\text{S, ABC}),$$

for they have equal bases PSQ, ABC, and the same height.

Thus the prism is divided into three pyr^ds of equal volume.

$$\therefore \textit{pyramid } (\text{S, ABC}) = \tfrac{1}{3}(\text{volume of prism})$$
$$= \tfrac{1}{3}(\textit{area of base}) \times (\textit{perp. height}).$$

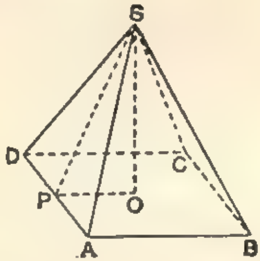
COROLLARY. A pyramid on a polygonal base may always be divided into a set of pyramids on triangular bases and having the same height as the given pyramid; hence

Volume of pyramid on any *base*

$$=\tfrac{1}{3}(\textit{area of base})\times \textit{height}.$$

EXERCISES.

1. Find (i) the slant surface, (ii) the volume of a right pyramid 15 cm. high, standing on a square base whose side is 16 cm.



Here $OS=15$ cm.
$AB=AD=16$ cm.
$OP=\frac{1}{2}AB=8$ cm.

From the $\triangle$ SOP, rt.-angled at O,
$SP^2=OS^2+OP^2=15^2+8^2=289$;
$\therefore\ SP=\sqrt{289}=17$ cm.

Area of face $SDA=\frac{1}{2}AD\times SP$
$=\frac{1}{2}(16\times 17)$ sq. cm.
$=136$ sq. cm.

Hence (i) *slant surface* $=\triangle SDA\times 4=544$ sq. cm.;
(ii) *volume* $=\frac{1}{3}$(*area of base*) $\times$ *height*
$=\frac{1}{3}(16^2\times 15)$ cu. cm.
$=1280$ cu. cm.

2. Find (i) the slant surface (to the nearest hundredth of a square inch), (ii) the volume of a right pyramid 7″ high, standing on a square base whose side is 6″.

3. Find the volumes of the pyramids in which
 (i) the base is a rectangle measuring 11 cm. by 7 cm., and the height is 12 cm.;
 (ii) the base is a triangle whose sides are 15 cm., 14 cm., 13 cm., and the height is 10 cm.

4. A right pyramid of height 9″ stands on a square base on a side of 8″. Find to the nearest hundredth of an inch (i) the slant height, (ii) the slant edge.

5. Find (i) the slant surface, (ii) the volume of a pyramid having the same base and height as a cube on an edge of 10 cm.

6. A right pyramid stands on a rectangular base whose sides are 24 cm. and 18 cm.; and each of the slant edges is 17 cm. Find the height and volume of the pyramid.

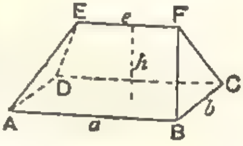
7. A right pyramid of height 2·4″ stands on a square base of which each side is 1·4″. Find (i) the cosine of the dihedral angle between each side-face and the base; (ii) the area of the projection of each side-face on the base.

8. The base of a right pyramid is an equilateral triangle on a side of 10 cm., and the vertical height is 5 cm. Find (i) the slant height, (ii) the area of one side-face, (iii) the cosine of the dihedral angle between the side-face and base.

Construct a plane angle having the same cosine, and measure it with your protractor.

9. Find to the nearest millimetre the height of a pyramid of which the volume is 270 cu. cm., and the base a regular hexagon on a side of 6 cm.

10. The solid shown in the diagram is called a wedge. Its base is a rectangle of length a and breadth b; the two ends are the $\triangle^s$ EAD, FBC; the remaining faces AEFB, DEFC are trapeziums having a common side EF which is thus par[l] to two sides of the base, and is called the *edge*.



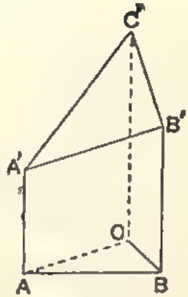
If the edge EF $= e$, and the height $= h$, prove that

$$\textit{volume of wedge} = \frac{hb}{6}\{2a + e\}.$$

[Draw planes through E and F perp. to the base, thus dividing the wedge into two pyramids and a prism. The two pyramids fit together into a single pyramid on a rectangular base of length $a - e$ and breadth b. The ends of the prism are the vertical sections, each of which has an area $\frac{1}{2}bh$, the length of the prism being e. Hence the volume of the wedge may be calculated.]

11. If in the last exercise the triangular faces of the wedge are equally inclined to the base, shew that the slant surface is given by the formula

$$\tfrac{1}{2}\{(a+e)\sqrt{4h^2+b^2} + b\sqrt{4h^2+(a-e)^2}\}.$$



12. The accompanying diagram represents an *oblique frustum*, or slice, formed by cutting a right triangular prism by a plane A′B′C′ not parallel to the base ABC.

If a, b, c are the lengths of the lateral edges AA′, BB′, CC′, prove that

$$\textit{volume of oblique frustum} = (\textit{area of base}) \times \tfrac{1}{3}(a+b+c).$$

[Cut the frustum by a plane parallel to the base through A′ the extremity of the least of the lateral edges, and thus divide the given solid into a pyramid and a prism.]

13. (EULER'S THEOREM). *In any polyhedron if* F, E, *and* V *denote the number of faces, edges, and vertices respectively, then* $E+2=F+V$.

[Suppose the polyhedron to be built up by fitting together n faces, one by one.

Beginning with *one* face, which has as many vertices as edges, we have $E=V$.

On adding the *second* face, there are *two* vertices and *one* edge in common with the first, so that the number of *new* edges is one more than the number of *new* vertices.

$$\therefore\ E=V+1.$$

The *third* face has *three* vertices and *two* edges in common with the former faces; and, as before, the number of new edges is one more than the number of new vertices.

$$\therefore\ E=V+2.$$

Proceeding in this way, step by step, when $n-1$ faces have been fitted together, we shall find

$$E=V+n-2.$$

In fitting on the last face no new edges or vertices are added, and finally $n=F$.

$$\therefore\ E=V+F-2,$$

$$\text{or } E+2=F+V.]$$

24. *There cannot be more than five regular polyhedra.*

Three plane angles at least are required to form a solid angle, and the sum of such plane angles must be less than 360° [Theor. 98]. It follows that each angle of the faces forming a regular polyhedron must be less than 120°. That is, the faces can only be *equilateral triangles*, *squares* or *pentagons*; for the angle of a regular hexagon is 120°, and any regular polygon of more than six sides has an angle greater than 120°.

Let D represent the number of degrees in a face-angle.

When the faces are *equilateral triangles*, $D = 60°$.

Then (i) $3D = 180°$, (ii) $4D = 240°$, (iii) $5D = 300°$, [$6D = 360°$].

Thus *three*, *four*, or *five* equilateral triangles, and not more than five, can be used to form a solid angle in a regular polyhedron.

When the faces are *squares*, $D = 90°$.

Then (iv) $3D = 270°$, [$4D = 360°$].

Thus *three* squares, and only three can be used.

When the faces are *pentagons*, $D = 108°$.

Then (v) $3D = 324°$, [$4D = 432°$].

Thus *three* regular pentagons, and only three can be used.

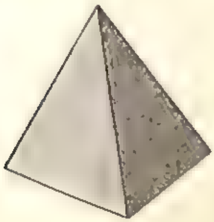
Hence there can only be five regular polyhedra.

25. If the plane faces which make up the surface of any of the regular polyhedra are supposed to be unfolded so that they all lie in one plane, we obtain in each case a plane figure made up of equilateral triangles, squares, or pentagons. Such a plane figure is called the net of the corresponding polyhedron.

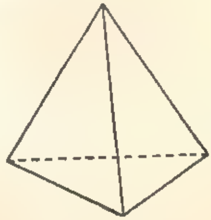
The regular polyhedra, and their nets (drawn on a reduced scale) are exhibited on pages 405–407.

THE REGULAR POLYHEDRA.

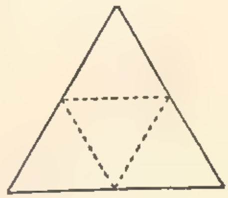
(i) The polyhedron of which each solid angle is formed by *three equilateral triangles* is called a regular tetrahedron.



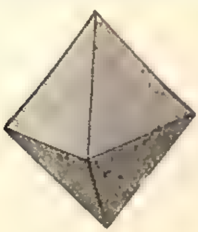
4 Faces
4 Vertices
6 Edges



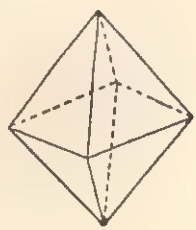
The net of the regular tetrahedron consists of four equal equilateral triangles placed as in the adjoining figure.



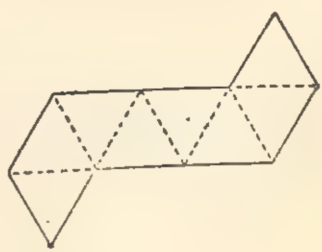
(ii) The polyhedron of which each solid angle is formed by *four equilateral triangles* is called a regular octahedron.



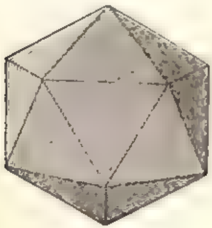
8 Faces
6 Vertices
12 Edges



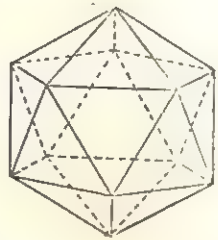
The net of the regular octahedron consists of eight equal equilateral triangles.



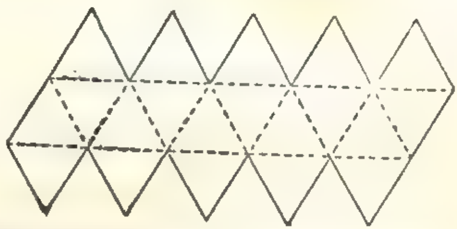
(iii) The polyhedron of which each solid angle is formed by *five equilateral triangles* is called a regular icosahedron.



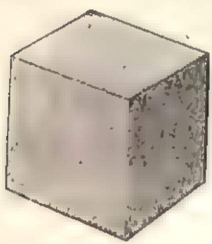
20 Faces
12 Vertices
30 Edges



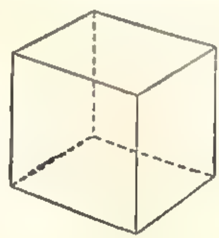
The net of the regular icosahedron consists of twenty equal equilateral triangles.



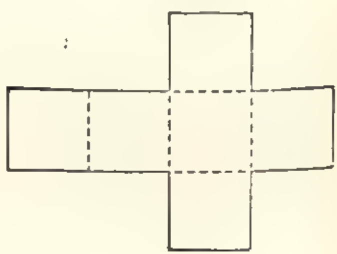
(iv) The regular polyhedron of which each solid angle is formed by *three squares* is called a cube.



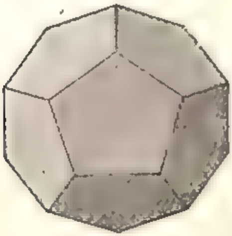
6 Faces
8 Vertices
12 Edges



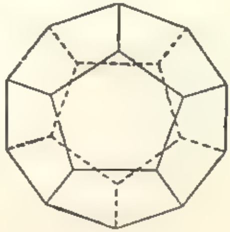
The net of the cube consists of six equal squares.



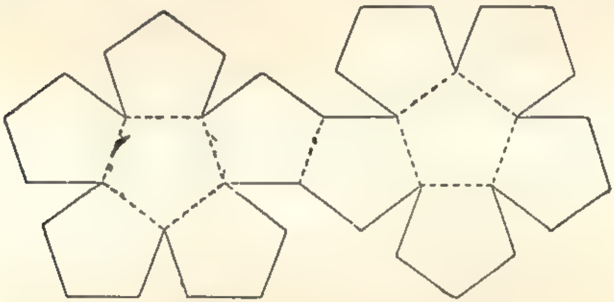
(v) The polyhedron of which each solid angle is formed by *three regular pentagons* is called a regular dodecahedron.



12 Faces
20 Vertices
30 Edges

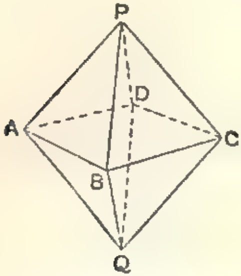


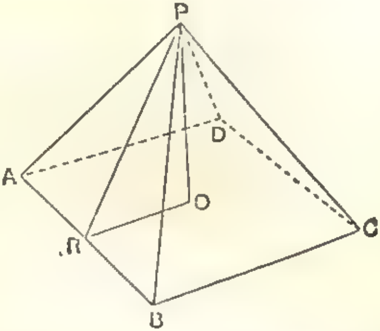
The net of the regular dodecahedron consists of twelve equal regular pentagons.



Models of the regular polyhedra may be made from their nets in the following manner. Draw the net on cardboard and cut it out along the outside lines, and cut partly through along the dotted lines. The faces may then be folded over till the edges come together, and the edges may be kept in position by using strips of gummed paper.

EXAMPLE. Find (i) the length of a diagonal, (ii) the surface, (iii) the volume, (iv) the dihedral angle of a regular octagon whose edge is $2m$.





It is evident from the diagram that a regular octahedron consists of two pyramids on opposite sides of a common square base ABCD. One of these pyramids is shewn on a larger scale in the right-hand figure.

PO is the perpendicular from P, O being the central point of the square ABCD. PR bisects AB at right angles.

Now $AB=2m$; $\therefore$ $RB=m$, and $RO=m$.

$$PR=RB\tan 60°=m\sqrt{3}.$$

From the rt.-angled $\triangle$ POR,

$$OP^2=PR^2-OR^2=3m^2-m^2=2m^2;$$

$$\therefore OP=m\sqrt{2}.$$

Hence (i) Diagonal of octahedron $=2OP=2m\sqrt{2}$.

(ii) Surface $=8\triangle PAB=8RP \cdot RB$

$$=8\times m\sqrt{3}\times m=8m^2\sqrt{3}.$$

(iii) Volume $=2$(vol. of pyramid whose vertex is P)

$$=2\times\tfrac{1}{3}OP\times(\text{area of base})$$

$$=2\times\tfrac{1}{3}m\sqrt{2}\times(2m)^2=\frac{8m^3\sqrt{2}}{3}.$$

(iv) Dihedral angle $=$ twice the $\angle$ PRO.

$$\text{Now }\tan PRO=\frac{PO}{OR}=\frac{m\sqrt{2}}{m}=\sqrt{2}=1{\cdot}414\,2\ldots.$$

Whence, by means of a Table of tangents, $\angle PRO=54°\ 44'$ approximately.

Thus the dihedral $\angle=109°\ 28'$ approximately.

MISCELLANEOUS EXERCISES ON POLYHEDRA.

1. The dimensions of a rectangular block being taken as 8·5 cm., 7·4 cm., 6·0 cm., find the volume. Supposing that each of the above measurements may be too great or too small by as much as 1 mm., find the greatest possible error in your result in excess and in defect; and express each as a percentage of the volume assumed in each case.

2. A deal plank 15″ wide is placed against the top of a wall 8 feet high, while the other end rests on the ground 6 feet from the wall. The thickness of the plank being $1\frac{1}{2}''$, find its weight, supposing 1 cubic foot of deal to weigh 56 lbs.

3. An equilateral triangle ABC, of which each side = 10 cm., is projected on a plane passing through AB; and the area of the projection is 34·64 sq. cm. What is the angle between the planes? [Calculate the cosine of the dihedral angle: then either use Tables, or construct and measure the corresponding plane angle. $\sqrt{3} = 1{\cdot}732\ldots$].

4. A right pyramid of height 8 cm. stands on a regular hexagonal base on a side of 4 cm. Find

(i) the *slant surface* to the nearest tenth of 1 sq. cm.

(ii) the *volume* to the nearest tenth of 1 cu. cm.

5. The base of a right pyramid is a square on a side of 6 cm., and the slant faces are equilateral triangles. Draw the net of the pyramid, and find its approximate height and volume.

6. The corners of the base of a right pyramid are at the points (9, 5, 0), (−9, 5, 0), (9, −5, 0), (−9, −5, 0); and its vertex is at the point (0, 0, 12). Find the slant surface. [See p. 382].

7. A right pyramid stands on a regular hexagon having a side of 5 cm., and its slant faces are inclined to the base at an angle of 60°. Find the volume.

8. If a perpendicular is drawn from a vertex of a regular tetrahedron on its base, shew that the foot of the perpendicular will divide each median of the base in the ratio 2 : 1.

9. Prove that the perpendicular from the vertex of a regular tetrahedron upon the opposite face is three times that dropped from its foot upon any of the other faces.

10. If p is the perpendicular drawn from a vertex of a regular tetrahedron to the opposite face, and if each edge $=2m$, shew that

$$3p^2=8m^2.$$

11. The length of each edge of a regular tetrahedron being $2m$, shew that

(i) the *whole surface* $=4m^2\sqrt{3}$; (ii) the *volume* $=\frac{2}{3}m^3\sqrt{2}$.

12. Find approximately the dihedral angle between any two adjacent faces of a regular tetrahedron.

13. Prove that (i) the sum of the squares on the four diagonals of a parallelepiped is equal to the sum of the squares on the twelve edges.

(ii) The sum of the squares on the edges of any tetrahedron is four times the sum of the squares on the straight lines which join the middle points of opposite edges.

14. In any tetrahedron the plane which bisects a dihedral angle divides the opposite edge into segments which are proportional to the areas of the faces meeting at that edge.

15. OA, OB, OC are conterminous edges of a cube, each of length a units. Prove that

(i) volume of pyramid (O, ABC) $=\frac{1}{6}\cdot a^3$;

(ii) area of triangle ABC $=\frac{\sqrt{3}}{2}\cdot a^2$;

(iii) perp. from O on plane ABC $=\frac{\sqrt{3}}{3}\cdot a$.

16. OA, OB, OC are lines in space, each perpendicular to the other two; and their lengths are a, b, c respectively. Prove that

(i) volume of pyramid (O, ABC) $=\frac{1}{6}\cdot abc$;

(ii) area of triangle ABC $=\frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$;

(iii) perp. from O on plane ABC $=abc/\sqrt{a^2b^2+b^2c^2+c^2a^2}$.

17. Shew how to cut a cube by a plane so that the lines of section may form a regular hexagon.

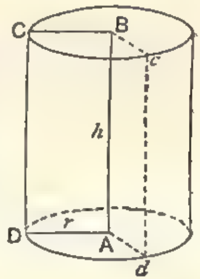
18. The greatest possible cube is cut from a right pyramid h inches high, standing on a square base whose side is a inches, one face of the cube being in the plane of the base of the pyramid. Prove that the edge of the cube $=ah/(a+h)$.

SOLIDS OF REVOLUTION.

THE CYLINDER.

26. DEFINITION. **A right circular cylinder** is a solid generated by the revolution of a rectangle about one of its sides as axis.

Thus if the rectangle ABCD revolves about AB as axis, the opposite side CD generates the curved surface of the cylinder represented in the diagram. The side CD, which moves parallel to the axis, is called the generating line of the surface; and since the sides AD, BC are always perpendicular to the axis, they move in parallel planes and describe circular ends or bases. The height of a right cylinder is the length of its axis AB.

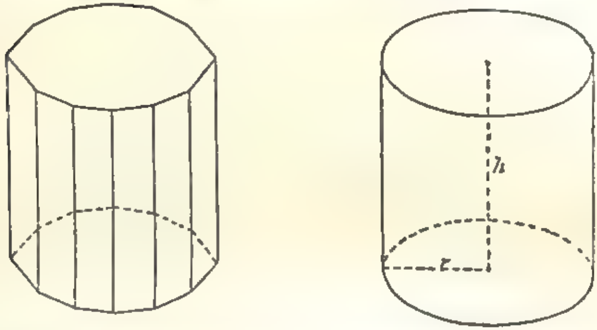


It is easily seen that every plane section of a right circular cylinder parallel to the base is a circle. Also every section parallel to the axis is a rectangle.

27. In general any surface described by a generating line which moves parallel to itself and slides continually over a fixed curve (not in the same plane with it), is said to be **cylindrical**. The fixed curve is called the **guide**; and in the particular case of a right circular cylinder the guide is a circle in a plane perpendicular to the generating line.

In the present Section, unless otherwise stated, only right circular cylinders will be considered.

28. *To find the* surface *and* volume *of a* cylinder.



Consider a right prism standing on a regular polygonal base. If the number of sides in the base is increased without limit, the polygon ultimately becomes a circle [Page 203], and the prism takes the form of a right cylinder. Thus a cylinder may be regarded as a prism in its limiting form; and accordingly expressions for the surface and volume of the cylinder follow from those given for the prism on p. 392.

Hence we have

(i) Curved surface *of cylinder* $=$ (*circumference of base*) $\times$ *height*

$$= 2\pi r \times h$$

$$= 2\pi rh \text{ units of area.}$$

(ii) Volume *of cylinder* $=$ (*area of base*) $\times$ *height*

$$= \pi r^2 \times h$$

$$= \pi r^2 h \text{ units of volume.}$$

NOTE 1. The *whole* surface $=$ curved surface $+$ area of ends

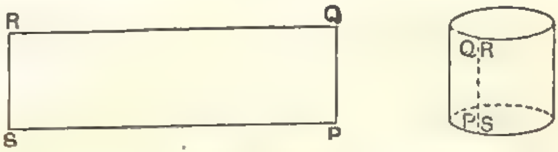
$$= 2\pi rh + 2\pi r^2$$

$$= 2\pi r(h+r).$$

NOTE 2. The volume of an *oblique* cylinder on any base, as of an oblique prism, is also given by the formula :—

Volume $=$ (*area of base*) $\times$ (*perp. height*).

NOTE 3. The formula for the curved surface of a cylinder may be illustrated thus.



Suppose the surface of the cylinder to be cut along a generating line PQ, and then unrolled on a plane; the surface will take the form of a rectangle PQRS, of which the length PS and breadth PQ are respectively the *circumference* and *height* of the cylinder.

Thus *curved surface* = PS × PQ

= *circumference* × *height.*

NOTE 4. Surfaces which can be unrolled (without stretching or tearing), and represented by plane figures are said to be developable.

EXERCISES.

[*In working examples which involve the use of* π, *the substitution of a numerical equivalent should be postponed as long as possible, and in each case the value should then be selected which will give a result of the required degree of accuracy.* See page 202.]

1. Find the curved surface (to the nearest square centimetre), and the volume (to the nearest cubic centimetre) of the cylinders whose dimensions are as follows:

(i) $r = 3{\cdot}0$ cm., $h = 8{\cdot}0$ cm. (ii) $r = 4{\cdot}5$ cm., $h = 7{\cdot}2$ cm.

2. Find to the nearest square centimetre the total surface of a cylinder whose height is 15·8 cm., and the diameter of whose base is 8·4 cm.

3. Find to the nearest cubic centimetre the volume of a cylinder, of height 12 cm., which just fits into a right prism standing on a square base whose side is 3·6 cm.

4. Find the locus of points at a given perpendicular distance from a given finite straight line.

If the given distance = 3·5 cm., and the length of the given straight line = 5·6 cm., find the surface of the locus to the nearest square centimetre.

EXERCISES ON CYLINDERS. (*Continued.*)

5. Find to the nearest square centimetre the whole surface of a hollow cylinder open at the ends, if the length is 12 cm., the external diameter 8 cm., and the thickness 2 cm.

6. The volume of a cylindrical column standing on a base whose diameter is 4 metres is 128·2 cubic metres; find its height to the nearest centimetre.

7. A cubic inch of gold is drawn into a wire 1000 yards long; find the diameter of the wire to the nearest thousandth of an inch.

8. The curved surface of a cylinder is 1000 sq. cm., and the diameter of its base is 20 cm.; find the volume of the cylinder. Also find its height to the nearest millimetre.

9. A column is formed by inserting a block of wood in a closely fitting cylindrical case, and then filling up the surrounding space with concrete. If the wood is in the shape of a right prism $8\frac{1}{4}$ ft. long on a rectangular base whose sides are 1 ft. 4 in. and 1 ft., find the required amount of concrete to the nearest cubic foot.

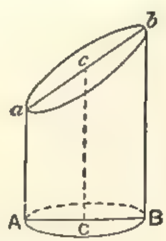
10. Find the weight to the nearest 100 grams of a cylindrical iron pipe 18 metres long, the external diameter being 5·4 cm., and the thickness 4 mm.; assuming that the specific gravity of iron is 7·79.

11. A copper wire 2 mm. in diameter is evenly wound about a cylinder whose length is 12 cm., and diameter 10 cm., so as to cover the whole surface. Find the length and weight of the wire assuming the specific gravity of copper to be 8·88.

12. The annexed diagram represents an *oblique frustum of a cylinder.* Suppose this solid cut by a plane through c parallel to the base AB, and hence prove that

(i) *Curved surface* $= 2\pi r \cdot \mathrm{C}c = 2\pi r \times \dfrac{h_1 + h_2}{2}$;

(ii) *Volume* $= \pi r^2 \cdot \mathrm{C}c = \pi r^2 \times \dfrac{h_1 + h_2}{2}$;

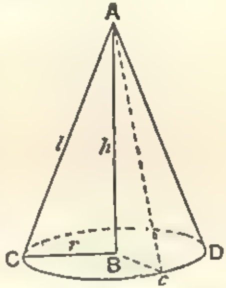


where h_1 and h_2 denote the greatest and least heights of the frustum.

THE CONE.

29. Definition. A right circular cone is a solid generated by the revolution of a right-angled triangle about one of the sides containing the right angle as axis.

Thus if the right-angled triangle ABC revolves about AB as axis, the hypotenuse AC generates the curved surface of the cone represented in the diagram. The hypotenuse AC, which in all its positions passes through the fixed point A, is called the generating line of the surface; and the circle described by the radius BC is the base of the cone.



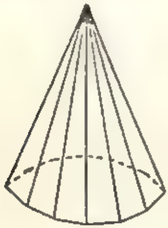
The point A is called the vertex, and the angle CAD (which is twice the angle A of the revolving triangle) is the vertical angle. The height of the cone is the length of the axis AB, and the slant height is the length of the hypotenuse AC.

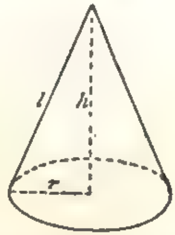
Every plane section of a right circular cone parallel to the base is a circle. Also every section of the cone through the vertex is a pair of intersecting straight lines.

30. In general any surface described by a generating line which passes through a fixed point and slides continually over a fixed guiding curve (not in the same plane with it) is said to be conical. In a right circular cone the guiding curve is a circle, and the vertex any point in the line through the centre of the circle normal to its plane.

In the present Section, unless otherwise stated, only right circular cones will be considered.

31. *To find the* **surface** *and* **volume** *of a* **cone.**





Consider a right pyramid standing on a regular polygonal base.

If the number of sides in the base is increased without limit, the polygon becomes a circle, and the pyramid ultimately takes the form of a right cone. Accordingly, expressions for the curved surface and volume of a cone follow from those given for the pyramid on pages 398, 401.

Thus, if h denotes the vertical height, l the slant height of the cone, and r the radius of the base,

(i) **Curved surface** *of cone* $= \frac{1}{2}$*(circumference of base)* $\times$ *(slant height)*

$$= \tfrac{1}{2} \times 2\pi r \times l$$

$$= \pi r l \text{ units of area.}$$

Again, since the volume of a pyramid is *one-third* that of the prism of the same base and height, it follows that the *volume of a cone* is *one-third that of the corresponding cylinder.* Hence

(ii) **Volume** *of cone* $= \frac{1}{3}$*(area of base)* $\times$ *(height)*

$$= \tfrac{1}{3} \times \pi r^2 \times h$$

$$= \tfrac{1}{3}\pi r^2 h \text{ units of volume.}$$

NOTE 1. The *whole* surface = curved surface + area of base

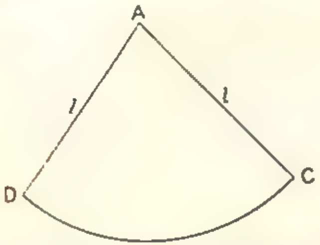
$$= \pi r l + \pi r^2$$

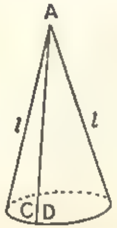
$$= \pi r (l + r).$$

NOTE 2. The volume of an *oblique* cone on any base, as of an oblique pyramid, is also given by the formula :

$$Volume = \tfrac{1}{3}\,(area\ of\ base) \times (vertical\ height).$$

NOTE 3. The formula for the curved surface of a cone may be illustrated thus.





Suppose the surface of the cone to be cut along a generating line AC, and then unrolled on a plane; the surface will take the form of a sector ACD, of which the radius AC and the arc CD are respectively the *slant height* and the *circumference of the base* of the cone.

Thus $\quad$ *curved surface* $= \frac{1}{2}$. *arc* CD $\times$ *radius* AC $\quad$ [Page 204]

$$= \frac{1}{2} \times 2\pi r \times l.$$

Hence we see that the surface of a right circular cone is developable.

EXERCISES.

1. If S is the surface, V the volume, and α the semi-vertical angle of a cone of height h, on a base of radius r, prove the following formulæ:

$$\text{(i)}\ S = \frac{\pi h^2 \tan \alpha}{\cos \alpha}; \quad V = \frac{1}{3}\pi h^3 \tan^2 \alpha.$$

$$\text{(ii)}\ S = \frac{\pi r^2}{\sin \alpha}; \quad V = \frac{1}{3} \cdot \frac{\pi r^3}{\tan \alpha}.$$

Hence prove that the volumes of cones with the same vertical angle are to one another as the cubes of their heights.

2. Find the surface to the nearest square centimetre, and the volume to the nearest cubic centimetre of the cones in which

(i) $r = 6$ cm., $l = 10$ cm. $\quad$ (ii) $r = 1{\cdot}2$ cm., $h = 3{\cdot}5$ cm.

3. Find to the nearest square centimetre the whole surface of a cone whose height is 40 cm., and the diameter of whose base is 18 cm.

4. Find to the nearest cubic centimetre the volume of a cone whose slant height and vertical height are 5·1 cm. and 4·5 cm. respectively.

[For further Exercises on Cones see p. 422.]

THE FRUSTUM OF A PYRAMID AND CONE.

32. Definition. A frustum (that is to say, a *slice*) of a pyramid or cone is the part cut off between the base and a plane parallel to the base.

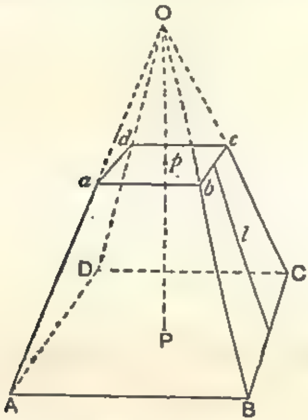


Fig. 1.

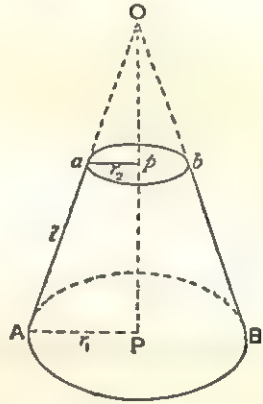


Fig. 2.

Thus in Fig. 1 the frustum of the pyramid (O, ABCD) is the part cut off between the base ABCD and the parallel section *abcd*.

In Fig. 2 the frustum of the cone (O, AB) is contained between the base AB and the parallel section *ab*.

In Fig. 1 the figures ABCD, *abcd*, and in Fig. 2 the figures AB, *ab* are called the ends of the frustum. The ends of the frustum of a pyramid are *similar figures* (Art. 13, p. 388). The ends of the frustum of a cone are *circles*.

The slant surface of the frustum of a pyramid is made up of *trapeziums*. If the base ABCD is *regular*, and the pyramid is *right*, these trapeziums are all equal.

33. Let the ends of the frustum contain E_1 and E_2 units of area, and let the perpendicular from the vertex O cut the ends at the points P and p respectively; then it has been proved (Art. 13, p. 388) that

$$E_1 : E_2 = OP^2 : Op^2.$$

34. *To find the* slant surface *and* volume *of the* frustum *of a right* pyramid *on a regular base of* n *sides.*

In Fig. 1 of the preceding page, let k denote the thickness Pp of the frustum, and let a_1, a_2 denote the lengths of any pair of corresponding sides BC, bc of the ends, and l the perpendicular distance between these sides, namely the *slant thickness* of the frustum. Also let the ends ABCD $abcd$ contain E_1 and E_2 units of area. Then

(i) **Slant surface** *of frustum* $= n$ times trapezium BCcb

$$= \tfrac{1}{2}(a_1 + a_2)l \times n \qquad \textit{Theor. } 28.$$

$$= \tfrac{1}{2}(na_1 + na_2)l \text{ units of area}$$

$$= \tfrac{1}{2}(\textit{sum of perimeters of ends}) \times (\textit{slant thickness}).$$

(ii) Denote the heights OP, Op by h_1 and h_2, so that $h_1 - h_2 = k$.

Now $\dfrac{E_1}{h_1^2} = \dfrac{E_2}{h_2^2} = m$, where m is some constant;

$$\therefore\ E_1 = mh_1^2, \text{ and } E_2 = mh_2^2.$$

Hence

Volume *of frustum* $=$ pyr^{d} (O, ABCD) $-$ pyr^{d} (O, $abcd$)

$$= \tfrac{1}{3}E_1h_1 - \tfrac{1}{3}E_2h_2$$

$$= \tfrac{1}{3}(h_1^3 - h_2^3)m$$

$$= \tfrac{1}{3}(h_1 - h_2)(h_1^2 + h_1h_2 + h_2^2)m$$

$$= \tfrac{1}{3}k(mh_1^2 + \sqrt{mh_1^2 . mh_2^2} + mh_2^2)$$

$$= \tfrac{1}{3}k[E_1 + \sqrt{E_1E_2} + E_2].$$

35. *To find the* **curved surface** *and* **volume** *of the* **frustum** *of a right* **cone.**

Considering the cone as the limiting form of a right pyramid on a regular polygonal base, we derive expressions for the curved surface and volume of a frustum of a cone from those given on the preceding page for the frustum of a pyramid.

In Fig. 2 of page 418 let r_1 and r_2 denote the radii of the ends AB and ab; let the thickness Pp be denoted by k, and the slant thickness Aa by l.

Then $E_1 = \pi r_1^2$, and $E_2 = \pi r_2^2$.

(i) **Curved surface** *of frustum of cone*

$$=\tfrac{1}{2}(\textit{sum of circumferences of ends}) \times (\textit{slant thickness})$$
$$=\tfrac{1}{2}(2\pi r_1 + 2\pi r_2)l$$
$$=\pi(r_1+r_2)l \text{ units of area.}$$

(ii) **Volume** *of frustum*
$$=\frac{k}{3}[E_1+\sqrt{E_1E_2}+E_2]$$
$$=\frac{k}{3}[\pi r_1^2+\sqrt{\pi r_1^2 \cdot \pi r_2^2}+\pi r_2^2]$$
$$=\frac{\pi k}{3}[r_1^2+r_1r_2+r_2^2] \text{ units of volume.}$$

NOTE 1. Since r_1+r_2 = twice the radius of a circular section equidistant from the two ends, we have

$$\text{Curved surface } \textit{of frustum} = \pi(r_1+r_2)l = 2\pi \cdot \frac{r_1+r_2}{2} \cdot l$$
$$=(\textit{circumference of } \text{mid-section}) \times (\textit{slant thickness}).$$

NOTE 2. If E_1, E_2 denote the areas of the ends, and M the area of the mid-section, then

$$\text{Volume } \textit{of frustum} = \frac{\pi k}{3}(r_1^2+r_1r_2+r_2^2) = \frac{\pi k}{6}(2r_1^2+2r_1r_2+2r_2^2)$$
$$=\frac{\pi k}{6}(r_1^2+(r_1+r_2)^2+r_2^2) = \frac{\pi k}{6}\left(r_1^2+4\cdot\left(\frac{r_1+r_2}{2}\right)^2+r_2^2\right)$$
$$=\frac{k}{6}(E_1+4M+E_2).$$

This last result is called the **prismoidal formula.** It is applicable to *any* solid whose ends are parallel (but not necessarily similar) figures of the same number of sides, each pair of corresponding sides in the two ends being parallel. Such a solid is called a **prismoid**; and frusta of a pyramid and cone are special cases of it.

EXERCISES.

On Frusta of Pyramids and Cones.

1. The ends of the frustum of a pyramid are squares on sides of 20 cm. and 4 cm.; if the frustum is 15 cm. in thickness, find its slant surface.

2. Find the curved surface of the frustum of a cone whose *slant* thickness is 5 cm., and whose circular ends are 8 cm. and 6 cm. in diameter.

3. Find the volume of the frustum of a pyramid, the ends being squares on sides of 8 cm. and 6 cm., and the thickness being 3 cm.

4. The *slant* thickness of a frustum of a cone is 5 cm., and the radii of its ends are 4 cm. and 1 cm. respectively, find its curved surface to the nearest sq. cm., and its volume to the nearest cubic cm.

5. The ends of the frustum of a pyramid are squares on sides of 8·0 cm. and 1·4 cm.; if the thickness of the frustum is 5·6 cm. find its slant surface to the nearest sq. cm.

6. The ends of the frustum of a pyramid are triangles, the sides of the base measuring 13 cm., 12 cm., 5 cm., and the sides of the top 6·5 cm., 6 cm., 2·5 cm.; if the thickness of the frustum is 8 cm., find the volume.

7. If r_1, r_2 are the radii of the ends of a frustum of a cone, whose height is h, shew that its volume is equal to the sum of the volumes of a cylinder and cone, each of height h, on bases of radii $\frac{1}{2}(r_1+r_2)$ and $\frac{1}{2}(r_1-r_2)$ respectively.

8. If the height of a frustum of a cone is equal to twice the mean proportional between the radii of its bases, shew that the slant side is equal to the sum of the radii.

9. A cone, whose height is n cm., is cut through by a plane parallel to the base and 1 cm. distant from it: express the volume of the frustum so formed as a fraction of the volume of the whole cone.

10. The frustum of a square pyramid is 6 cm. thick, and the area of one end is four times the area of the other. If the volume is 350 cubic centimetres, find the dimensions of the ends.

EXERCISES.

(*Miscellaneous Examples on Cones.*)

1. ABC is a triangle in which $a=6{\cdot}5$ cm., $b=2{\cdot}0$ cm., and the perpendicular from C on AB $=1{\cdot}6$ cm. Find to the nearest cubic centimetre the volume of the double cone formed by a complete revolution of the triangle about the side AB.

2. Find to the nearest foot the length of canvas, 1 yard wide, which will be required to make a conical tent 30 ft. in perpendicular height, and covering 1386 sq. ft. of ground.

3. Find to the nearest tenth of a metre the height of a conical tent which stands on a circular base of diameter 8·0 cm. and which contains 90·478 cubic metres of air.

4. Find to the nearest sq. cm. the *whole* surface of the greatest cone that can be cut from a solid cube whose edge is 20 cm., the base of the cone being in the same plane as the base of the cube.

5. Water flows at the rate of 10 m. per minute from a cylindrical pipe 5 mm. in diameter. How long would it take to fill a conical vessel whose diameter at the surface is 40 cm. and depth 24 cm.?

6. A cone is cut by a plane parallel to the base and the upper portion is removed. If the remainder is $\frac{8}{9}$ of the surface of the whole cone, find the ratio of the segments into which the cone's altitude is divided by the plane.

7. From a solid cylinder whose height is 2·4 cm. and diameter 1·4 cm., a conical cavity of the same height and base is hollowed out. Find the whole surface of the remaining solid to the nearest square centimetre.

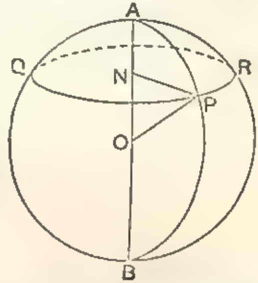
8. A conical vessel 7·5 cm. deep and 20·0 cm. across the top is completely filled with water. If sufficient water is now drawn off to lower its level by 6·0 cm. find the surface of the vessel thus exposed, to the nearest square millimetre.

9. Two conical vessels with a common vertex and axis are placed with the vertex downwards and the axis vertical. The inner cone is filled with oil, and the remainder of the outer with water. If the diameters at the surfaces of oil and water are 7·0 cm. and 11·2 cm. respectively, find the ratio of the weights of the oil and water, assuming the specific gravity of the oil to be 0·92.

10. Into each end of a solid cylinder, whose length is 10 cm. and diameter 8 cm., a conical cavity is bored; if the diameter of each cavity is 6 cm. and its depth 4 cm., find the whole surface of the remaining solid to the nearest square centimetre.

THE SPHERE.

36. DEFINITION. A **sphere** is a solid generated by the revolution of a semi-circle about its diameter as axis.



Thus if the semi-circle APB revolves about the diameter AB, the semi-circumference APB describes the surface of a sphere. And since, as the semi-circumference revolves, all points in it remain at a constant distance from its centre O, it follows that *the surface of a sphere is the locus in space of all points whose distance from a fixed point is constant.*

The fixed point is called the **centre**, and the constant distance the **radius** of the sphere. A **diameter** is any straight line through the centre terminated both ways by the surface. Thus *all diameters are equal.*

37. *Every plane section of a sphere is a circle.*

In the above Figure let the plane QPR cut the sphere whose centre is O, and whose radius is r; let P be any point on the line of section.

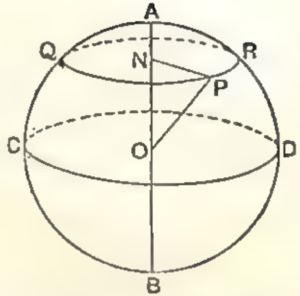
Draw ON perp. to the cutting plane, and let p be the length of ON. Join OP, PN.

Then since ON is perp. to NP in the plane QPR,

$$\therefore\ PN^2 = OP^2 - ON^2$$
$$= r^2 - p^2;$$
$$\therefore\ PN = \sqrt{r^2 - p^2} = \text{a constant.}$$

$\therefore$ the locus of P is a circle whose centre is the fixed point N.

DEFINITION. The diameter AB perpendicular to a plane section QPR is said to be the **axis** of the section, and the extremities A, B are called its **poles**.



38. If the plane section passes through the centre, N coincides with O, and the radius of the circle QPR assumes its greatest length, being then equal to the radius of the sphere.

The line of section of a sphere by a central plane is called a **great circle**; all other plane sections are called **small circles.**

39. In a sphere of radius r, if r_1 is the radius of any plane section whose distance from the centre is p, it has been shewn that $r_1 = \sqrt{r^2 - p^2}$. Hence if the cutting plane moves parallel to itself away from the centre O, then as p increases, r_1 will decrease. Thus *the radius of a plane section of a sphere continually diminishes as the distance of the plane from the centre increases*; and ultimately when N coincides with A, r_1 vanishes, so that the plane meets the sphere at A only, and is then said to be a **tangent plane** at that point. It follows that *at any point on a sphere there is* ONE *tangent plane, namely the plane perpendicular to the radius through that point.*

40. Any straight line drawn on a tangent plane through its point of contact will meet the surface at one point only, and is said *to touch* the sphere at that point. Thus at any point on a sphere there may be an infinite number of **tangent lines**, each being perpendicular to the radius through that point; and a tangent plane may be generated by rotating a tangent line about the radius through its point of contact.

41. *If* AB *is a diameter of a sphere, there is* ONE *great circle having* AB *as axis, and an infinite number of great circles passing through the poles* A *and* B.

42. *Through any* TWO *given points on a sphere (not the extremities of a diameter), one, and only one,* GREAT *circle can be drawn*; for the two given points and the centre of the sphere determine one, and only one, central cutting plane.

NOTE. The minor arc of a *great circle* through two given points on a sphere is called their spherical distance. It will be shewn (p. 437) that this arc is the shortest line that can be drawn on the surface between the two points. Now since all great circles of a sphere are equal, an arc of any great circle is measured by the angle which it subtends at the centre. [Theorem 69].

Thus in the diagram the spherical distance between the points Q and C is measured by the angle QOC and estimated in degrees. The spherical distance of any point on the great circle CD from its pole A is 90°.

43. *Through any* THREE *given points on a sphere one, and only one, circle (not necessarily great) can be drawn on the surface*; for the three points determine a plane which cuts the sphere in a circle passing through those points.

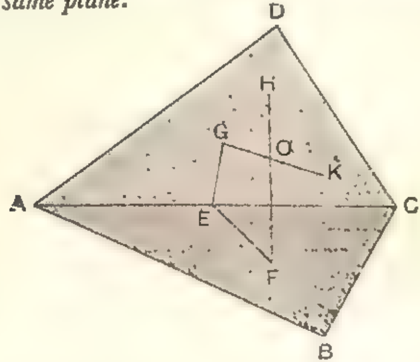
44. *An infinite number of spheres can pass through* TWO *given points, and their centres lie in a* FIXED PLANE.

For if Q and R are the given points, it is clear that the locus of points equidistant from Q and R is the plane which bisects the join QR at right angles. Hence with any point in this plane as centre a sphere may be drawn through Q and R.

45. *An infinite number of spheres can pass through* THREE *given points, and their centres lie on a fixed* STRAIGHT LINE.

For in the Figure of the preceding page, let P, Q, R be any three points, and N the centre of the circle through them. Let AB be the perpendicular to the plane of P, Q, R through N. Then if O is *any* point in AB, it is easily proved that the $\triangle^s$ OPN, OQN, ORN are congruent; so that OP = OQ = OR. Thus with any point in AB as centre a sphere may be drawn through P, Q, R. In other words, an infinite number of spheres can pass through P, Q, R; and the locus of their centres is the straight line AB, namely the perpendicular to the plane of the $\triangle$ PQR through its circum-centre.

46. *One sphere, and only one, can pass through any* FOUR *points not in the same plane.*



Let A, B, C, D be four points not in the same plane, and let F and G be the circum-centres of the $\triangle^s$ABC, ADC.

Let FH, GK be the perps. to the planes ABC, ADC through F and G respectively.

Then every point in FH is equidistant from A, B, C ; and every point in GK is equidistant from A, D, C ; hence every point in FH and in GK is equidistant from A and C.

But the locus of points equidistant from A and C is the plane which bisects AC at right angles ;

∴ FH and GK both lie in this plane ; and since they cannot be par[l] (being perp[s] to intersecting planes) they must meet at some point O.

Then O, the only point common to FH and GK, is equidistant from A, B, C, and D.

∴ a sphere having its centre at O, and radius OA, will pass through, A, B, C, and D ; and this is the only sphere that can pass through the four given points.

EXERCISES ON THE SPHERE.

(Theoretical.)

1. Shew that any tangent plane to the inner of two concentric spheres will cut the outer in a circle of constant radius.

2. Find the locus of points on a sphere at a given distance from a given point P. Illustrate by diagrams the several cases when P is inside, on, or outside the sphere.

3. Two spheres of radius r and r' have their centres a cm. apart. What are the conditions that the spheres (i) touch (ii) cut one another?

If the spheres cut, prove that their line of section is a circle.

4. How many tangent lines may be drawn to a sphere from an external point? What surface do such tangents generate? Prove that they are all equal; and find the locus of their points of contact.

5. If a fixed sphere is cut by planes which pass through a given point, find the locus of the centres of the sections. Distinguish between the cases which arise when the given point is (i) within, (ii) on, (iii) outside the fixed sphere.

6. A fixed point O is joined to any point P in a given plane not containing O; and on OP a point Q is taken such that OP . OQ = constant. Find the locus of Q.

7. Shew how a sphere may be inscribed in any tetrahedron, so as to be touched by each of its faces. Shew also that four spheres may be escribed to a tetrahedron.

8. If R and r denote the radii of the spheres circumscribed about, and inscribed in, a regular tetrahedron each edge of which measures $2a$, shew that

$$R = 3r = \frac{a}{2}\sqrt{6}.$$

9. Find the locus of points in a given plane at which a straight line of given length and fixed position in space subtends a right angle.

10. If a sphere can be placed in a wire tetrahedron so as to touch all the edges, the sum of each pair of opposite edges is the same.

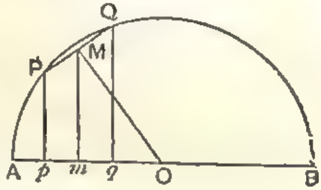
47. *To find the* surface *of a* sphere.

Let O be the centre of a sphere of radius r, generated by the revolution of a semi-circle APB about the diameter AB.

Let PQ be a side of half a regular polygon (of an even number of sides) inscribed in the semi-circle.

Draw OM perp. to PQ and therefore bisecting it.

Draw Pp, Mm, Qq perp. to AB.



Then as the semi-circle revolves about AB, the side PQ will generate the curved surface of a conical frustum ;

$\therefore$ curved surface of frustum $= 2\pi \,.\, \text{M}m \times \text{PQ}$. (Art. 35, Note 1.)

Now, if θ denotes the angle between pq and PQ,

$$pq = \text{PQ}\cos\theta = \text{PQ}\,.\,\frac{\text{M}m}{\text{MO}},$$

for Mm and MO are respectively perp. to pq and PQ ;

$$\therefore\ \text{M}m\,.\,\text{PQ} = \text{OM}\,.\,pq.$$

$\therefore$ curved surface of frustum $= 2\pi\,.\,\text{OM} \times pq$.

And when the number of sides is indefinitely increased, and PQ in consequence indefinitely diminished, the surface of this frustum ultimately becomes the belt of the sphere generated by the *arc* PQ. Also in the limit, $\text{OM} = r$.

$\therefore$ the area of belt $= 2\pi r \times$ (projection of PQ on AB).

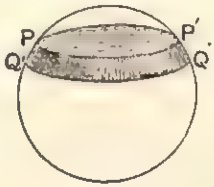
But the surface of the sphere is the sum of all the belts corresponding to the successive sides ;

and the sum of the projections of all the sides $= \text{AB} = 2r$.

$$\therefore\ \textit{the surface of sphere} = 2\pi r \times 2r$$
$$= 4\pi r^2.$$

NOTE. The surface of a sphere is thus *four times the area of a great circle.*

48. Definitions. A **frustum of a sphere** is the part cut off between two parallel planes. The curved surface of a frustum of a sphere is called a **zone**.



Either part of a sphere cut off by a single plane is called a **segment**. The curved surface of a spherical segment is sometimes called a **cap**.

If in a frustum of a sphere one of the planes of section PP′ moves parallel to itself until it becomes a tangent plane (Art. 39.) the circular end will vanish, and the frustum becomes a segment.

49. From Art. 47

$$\textit{the area of a } \text{zone} = 2\pi r \times (\textit{distance between planes})$$
$$= 2\pi rk,$$

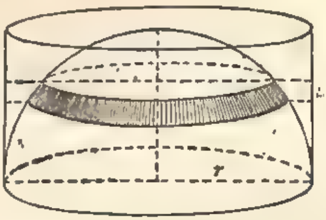
where r is the radius of the *sphere*, and k the thickness of the frustum.

This was proved when the thickness was indefinitely small; but, by addition of narrow zones, the formula holds good for all values of k.

Similarly *the curved surface of a* segment $= 2\pi rh$, where r is the radius of the sphere, and h the height of the segment.

Note 1. Since the area of a zone depends only on the radius of the *sphere* and the thickness of the frustum, it follows that *a zone of given thickness has the same area from whatever part of the sphere it is cut.*

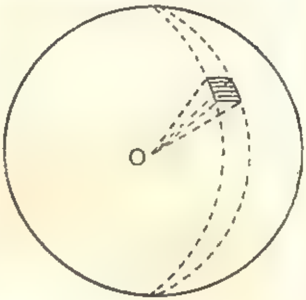
Note 2. If a cylinder is circumscribed about a sphere, and a frustum cut by two planes perpendicular to the axis of the cylinder, then *the area of the zone is equal to the area of the corresponding cylindrical belt,* since each area $= 2\pi rk$.



Hence *the whole surface of the sphere is equal to the curved surface of the circumscribed cylinder.*

50. *To find the* **volume** *of a* sphere *of radius* r.

The surface of the sphere may be divided into minute portions or elements of area, as indicated in the diagram. If these elements of area are diminished indefinitely, each one tends to become ultimately plane, and may be considered the base of a pyramid, whose vertex is at the centre, and whose height is the radius of the sphere.



Now the volume of any such pyramid

$$=\tfrac{1}{3}(\text{the element of surface})\times r.$$

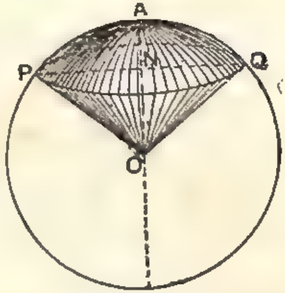
But the sum of all the elements of surface is the whole surface of the sphere;

and the sum of all the corresponding pyramids is the whole volume of the sphere.

$$\therefore\ \textit{volume of sphere}=\tfrac{1}{3}(\textit{surface of sphere})\times r$$
$$=\tfrac{1}{3}\times 4\pi r^2\times r$$
$$=\tfrac{4}{3}\pi r^3$$

51. Definition. A **sector** of a sphere is the solid bounded by a segmental cap and the conical surface traced out by a radius of the sphere moving round the edge of the cap. (See Fig. p. 431.)

52. By the method of Art. 50 it may be shewn that *the volume of a* **sector** *of a sphere is* $\tfrac{1}{3}Sr$, *where* S *denotes the surface of the segmental cap.*



53. The **volume** of the segment PAQ may be found by taking the difference of the solid sector (O, PAQ) and the cone (O, PQ).

Let r denote the radius of the sphere, r_1 the radius PN of circular base, and h the height AN of the segment.

Then $$\textit{volume of segment} = \frac{r}{3} \cdot 2\pi r h - \frac{1}{3}\pi r_1^2 (r-h)$$

$$= \frac{\pi}{3}\{2r^2 h - r_1^2(r-h)\}. \quad \text{.........(i)}$$

But by Theorem 57, $$h(2r-h) = r_1^2. \quad \text{.........(ii)}$$

To obtain the required volume in terms of r and h, substitute in (i) the value of r_1^2 obtained from (ii);

and to obtain the volume in terms of r_1 and h, substitute in (i) the value of r from (ii).

Simplifying in each case, it will be found that

$$\textit{volume of segment} = \pi h^2\left(r - \frac{h}{3}\right) \quad \text{.........(iii)}$$

$$= \frac{\pi h}{6}(3r_1^2 + h^2). \quad \text{.........(iv)}$$

54. The **volume** of a **frustum** of a sphere of thickness k may be found by taking the difference between two segments of heights h_1 and h_2 respectively, and remembering that $h_1 - h_2 = k$.

Use the result marked (iii), and on reduction by means of (ii), it will be found that

$$\textit{volume of frustum of sphere} = \frac{\pi k}{6}(3r_1^2 + 3r_2^2 + k^2),$$

where r_1 and r_2 are the radii of the circular ends.

7

EXERCISES ON THE SPHERE.

(Numerical.)

1. Find to the nearest square centimetre the surfaces of the spheres whose radii are (i) 2·4 cm., (ii) 10·5 cm. Also find their volumes to the nearest cubic centimetre.

2. Find to the nearest penny the cost of gilding a hemispherical dome 12 m. in diameter at 1s. 6d. per square metre.

3. Find the radius of a sphere whose surface is equal to that of a circle 2·8 cm. in diameter.

4. How many solid spheres 6 cm. in diameter could be moulded from a solid metal cylinder whose length is 45 cm. and diameter 4 cm.?

5. Find to the nearest cubic centimetre the volume of a spherical shell whose internal and external radii are 6 cm. and 5 cm. respectively.

6. Find to the nearest square centimetre the whole surface of a hemispherical bowl, 1 cm. in thickness, and 10 cm. in external diameter.

7. A solid metal sphere, 6 cm. in diameter, is formed into a tube 10 cm. in external diameter and 4 cm. in length; find the thickness of the tube.

8. Find the whole surface and weight of a hemispherical copper bowl 12 cm. in external diameter and 1 cm. in thickness; assuming that 1 cu. cm. of copper weighs 8·88 grams.

9. A sphere whose radius is 3·5 cm. is enclosed in a hollow cylinder of the same radius, whose length is equal to its circumference: how many cubic centimetres are there in the remaining part of the cylinder?

10. Find to the nearest millimetre the radius of a sphere whose surface is equal to the whole surface of a cylinder of height 16 cm. and diameter 4 cm.

11. Assuming a drop of water to be spherical, and one tenth of an inch in diameter, to what depth will 500 drops fill a conical wine glass, the cone of which has a height equal to the diameter of its rim?

12. A sphere of diameter 6 cm. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cm. If the sphere is completely submerged, by how much will the surface of the water be raised?

13. The weights of two spheres are in the ratio of 8 to 17, and the weights of a cubic foot of the material in the two spheres are in the ratio of 289 to 64: compare their radii.

14. Find the approximate weight of 50,000 spherical lead bullets, each 8 mm. in diameter, the specific gravity of lead being 11·35.

15. Assuming the specific gravity of copper to be 8·88, find the weight of a spherical copper shell, 12 cm. in external diameter, and 2 cm. thick.

16. How many metres of wire 0·4 mm. in diameter may be drawn from the amount of copper required to mould a solid sphere of diameter 18 cm.?

If the diameter of the wire is decreased by 5 per cent., by how much per cent. will its length be increased?

17. Find the *whole* surface and volume of a spherical segment greater than a hemisphere, its height being 18 cm., and the radius of the *sphere* being 13 cm.

18. Find the whole surface and volume of a spherical frustum, the diameter of the sphere being 20 cm., and the distance of the plane ends from the centre (on the same side) being 6 cm. and 8 cm.

19. Find the area of a spherical zone, the radii of the two ends being 12 cm. and 5 cm., and the thickness 7 cm.

20. A sphere of diameter 24 feet is placed so that its centre is 37 feet from the observer's eye. Find the area of that part of the sphere's surface that is visible to the observer.

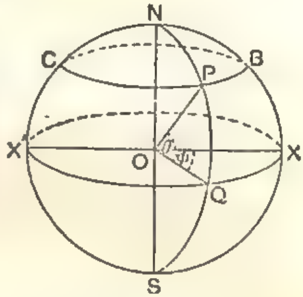
21. Considering the Earth as a sphere of diameter 8000 miles, find roughly in feet at what height above the ground would one-millionth of the Earth's surface be visible.

22. AB is an arc of a circle whose centre is O. If the sector OAB rotates about the radius OA, shew that the curved surface of the solid generated is expressed by $\pi(\text{chord AB})^2$.

23. A cylinder is circumscribed about a hemisphere, and a cone is inscribed in the cylinder so as to have its vertex at the centre of one end, and the other end as base; shew that

$$\frac{\text{vol. of cylinder}}{3}=\frac{\text{vol. of hemisphere}}{2}=\frac{\text{vol. of cone}}{1}.$$

LINES OF REFERENCE ON A SPHERE. LATITUDE AND LONGITUDE.



55. In the sphere whose centre is O a diameter NS is taken as axis. The great circle XQX′, of which NS is the axis, is then called the equator, and the points N and S are respectively its North and South poles.

Then NS will be the axis, and N, S the poles, of all small circles in planes parallel to the equator. Let BPC be any such small circle, and P any point on it.

Through the poles N and S an infinite number of great circles may be drawn: let NPS be that which passes through P, and let it cut the equator at Q.

Now it may readily be proved (as in Art. 37) that the ∠NOP is constant for all points on the small circle BC; hence the arc NP, called the North **polar distance** of P, is constant for all such points. And since the ∠NOQ = 90°, the ∠POQ is also constant.

The ∠POQ, or the corresponding arc PQ is called the **latitude** of the point P, and may be defined as its *spherical distance from the equator*.

The small circle BPC is called a **parallel** of latitude, for the latitude of all points on it is the same. This latitude is denoted in the diagram by θ.

Of the system of great circles which pass through the poles N and S, semi-circumferences, such as NPQS, are called **meridians**.

The position of a meridian relative to some fixed meridian of reference is determined by the angle between their planes. Thus if NXS is the fixed meridian of reference, the position of the meridian NPQS is determined by the ∠XOQ, for OX and OQ are perpendicular to the line of section NS.

The $\angle$XOQ, denoted in the diagram by ϕ, is called the longitude of the meridian NPQS, or the longitude of all points on that meridian.

It is now clear that the position of a point P on the surface of a sphere is fixed if we know on what *parallel of latitude* and on what *meridian of longitude* it lies: briefly, if we know its latitude and longitude. These angular data correspond to the linear coordinates which fix the position of a point on a plane.

EXERCISES.

1. A plane section of a sphere is taken in latitude θ. If the radius of the sphere is r, and the radius of the circular section r_1, shew that

$$r_1 = r\cos\theta.$$

2. Taking the equatorial radius of the Earth as 3960 miles, find approximately, using Four-Figure Tables,

(i) the length of the equator;

(ii) the length of a *knot* (1 minute of equatorial arc);

(iii) the length of the parallel of latitude 55°;

(iv) through how many miles an hour London moves in consequence of the rotation of the Earth [Latitude of London = 51° 30′].

3. A segment is cut off from a sphere of radius r by a plane in latitude θ; deduce from the formulæ of Arts. 49 and 53.

(i) Surface of segmental cap $= 2\pi r^2(1-\sin\theta)$.

(ii) Volume of segment $= \frac{1}{3}\pi r^3(1-\sin\theta)(2-\sin\theta-\sin^2\theta)$.

4. Shew that the area of a spherical zone included between the latitudes θ_1 and θ_2 is given by the formula

$$2\pi r^2(\sin\theta_1 - \sin\theta_2).$$

5. Taking the mean diameter of the Earth as 7922 miles, find as nearly as you can with Four-Figure Tables

(i) the whole surface of the Earth;

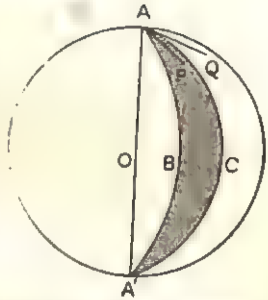
(ii) the surface of the Arctic Cap [lat. of Arctic Circle $= 66\frac{1}{2}°$];

(iii) the surface of the Tropical Zone [lat. of Tropics $= 23\frac{1}{2}°$ N. and S.].

6. If θ and ϕ are the latitude and longitude of a point (x, y, z) on the surface of a sphere of radius r, shew that

$$x = r\cos\theta\cos\phi, \quad y = r\cos\theta\sin\phi, \quad z = r\sin\theta.$$

LUNES OF A SPHERICAL SURFACE.



56. Any two planes cutting a sphere centrally must intersect in a diameter; hence any two great circles must cut at the extremities of a diameter.

The angle at which two great circles cut one another is called a **spherical angle**, and is measured by the angle between the tangents to the circles at either point of intersection.

Thus the angle between the great circles ABA′, ACA′ is that between the tangents AP, AQ. Now these tangents are respectively in the planes of the circles, and they are perpendicular to the line of section AA′; therefore the angle between them is the measure of the dihedral angle between the planes.

Hence *a spherical angle is measured by the dihedral angle between the planes of the intersecting great circles.*

57. Definition. A **Lune** of a sphere is a part of the surface cut off by two planes passing through a diameter. The boundaries of a lune are therefore two great semi-circles; and the spherical angle between them is called the **angle of the lune.**

58. *To find the* **area** *of a* **lune.**

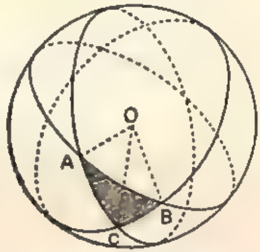
It will easily be seen that the areas of lunes are proportional to their spherical angles; moreover the whole surface of a sphere may be regarded as a lune whose angle is 360°. Hence, if the angle of a lune is D degrees,

$$\textit{Area of lune} = (\textit{whole surface of sphere}) \times \frac{D}{360}$$

$$= \pi r^2 \cdot \frac{D}{90}.$$

SPHERICAL TRIANGLES.

59. DEFINITION. A triangle drawn on the surface of a sphere, the sides being the arcs of *great circles*, is called a spherical triangle.



60. If the vertices of a spherical triangle ABC are joined to the centre O of the sphere, the three planes AOB, BOC, COA form at the point O a trihedral angle closely related to the triangle ABC.

For instance: the *sides* of the spherical triangle ABC may be considered either as the arcs AB, BC, CA, or as the *face-angles* AOB, BOC, COA. Thus the sides *a, b, c* of a spherical triangle may be measured in degrees.

Again, the spherical angles A, B, C have the same measure as the dihedral angles of the solid angle (O, ABC).

From this connection between a spherical triangle and a trihedral angle we conclude that:

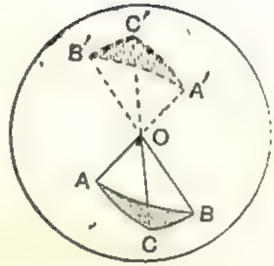
(i) *The sum of any two sides of a spherical triangle is greater than the third.*

For the sum of the face-angles AOB, AOC, which measure the sides AB, AC, is greater than the third face-angle BOC, which measures the side BC. (*Theor.* 97.)

(ii) *The sum of the sides of a spherical triangle is less than the circumference of a great circle.*

For the sum of the face-angles AOB, BOC, COA is less than four right angles (*Theor.* 98); hence the sum of the corresponding arcs is less than four quadrants.

NOTE. From the first of these results we see that *any side of a spherical polygon is less than the sum of the remaining sides.* And hence: *The shortest line on the surface of a sphere between two points is the minor arc of the great circle which passes through them*; for any other line may be regarded as the sum of minute arcs of great circles, in the limit when each arc is diminished indefinitely.



61. The diameters AA′, BB′, CC′ through the vertices of a spherical triangle ABC meet the surface at the vertices of a spherical triangle A′B′C′, which is said to be the opposite or symmetric of the first.

62. Now just as the trihedral angles (O, ABC), (O, A′B′C′) are equal in their several parts, but yet are not *superposable* [see page 377], so the sides and angles of the spherical triangle ABC are severally equal to the sides and angles of its *opposite* A′B′C′; but owing to the curvature of their faces the two triangles are not in general superposable.

For if we look on the convex face of each triangle, the sequence of the vertices A, B, C is clockwise, while that of the corresponding vertices A′, B′, C′ is counter-clockwise.

Now this disparity, if the triangles were *plane*, might be met by turning one of them over before superposition [see Part I. p. 19]; but such a reversal in the case of spherical triangles would bring the two convex faces into contact, so that coincidence would be impossible.

NOTE. An *isosceles* spherical triangle and its opposite are superposable. For, A and A′ being the vertices, we should have

$$AB = AC = A'B' = A'C';$$

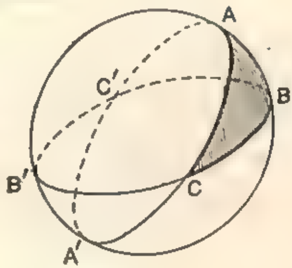
so that by placing A′ on A, B′ might be made to fall on C, and C′ on B.

63. Though a spherical triangle and its opposite are not generally superposable, they are always *equal in area.*

For the triangle ABC may be divided into minute triangles, each of which will have its opposite in the triangle A′B′C′. Now each such pair of opposites, if taken small enough, may be regarded as *plane* and therefore superposable. Thus the triangle ABC is made up of elements of area each of which has its equal in the triangle A′B′C′.

NOTE. Just as a plane triangle may be divided into three isosceles triangles by joining the vertices to the circumcentre, so a spherical triangle ABC may be divided into three isosceles spherical triangles by joining the vertices to *the pole of the plane* ABC. Now each isosceles triangle has for its opposite an equal and superposable *isosceles triangle;* and the three opposites make up the whole opposite triangle A′B′C′.

64. *To find the* area *of a* spherical triangle.



Let Δ denote the area of the spherical triangle ABC, formed by arcs of the great circles ABA′, BCB′, CAC′; the points A′, B′, C′ being respectively opposite to A, B, C.

Now

$$\text{lune of } \angle A = \Delta + \triangle A'BC;$$
$$\text{lune of } \angle B = \Delta + \triangle AB'C;$$
$$\text{lune of } \angle C = \Delta + \triangle ABC' = \Delta + \triangle A'B'C; \quad \text{(Art. 63.)}$$

$\therefore$, by addition,

$$\text{lune } A + \text{lune } B + \text{lune } C = 2\Delta + \{\Delta + \triangle A'BC + \triangle AB'C + \triangle A'B'C\}$$

or, $$\frac{\pi r^2}{90}\{A+B+C\} = 2\Delta + \text{surface of hemisphere};$$

or, $$\frac{\pi r^2}{180}(A+B+C) = \Delta + \pi r^2;$$

$$\therefore \ \Delta = \frac{\pi r^2}{180}\{A+B+C-180°\}. \quad \text{.................(i)}$$

NOTE. Since Δ must be a positive quantity, we see from (i) that $A+B+C$ must be greater than 180°; that is, *the sum of the angles of the spherical triangle is greater than two right angles.* The angle $A+B+C-180°$ is called the spherical excess, and is denoted by E.

Hence $$\Delta = \frac{\pi r^2}{180} \times E, \text{ if E is measured in degrees;}$$

or $$\Delta = r^2 \times E, \text{ if E is given in radian measure.}$$

MISCELLANEOUS EXERCISES.

(*In some of the following examples numerical work may be shortened by the use of Four-Figure Logarithms.*)

1. A cube and a sphere being of equal volume, find the ratio of the radius of the sphere to the side of the cube.

2. The diagonal of a cube is 58·4 cm.; find the radius of a sphere whose surface is equal to that of the cube.

3. Find the surface and volume of a cone whose base is 97·6 sq. cm., and such that the height is to the radius of the base as 11 to 6.

4. A solid, consisting of a right cone standing on a hemisphere, is placed upright in a right cylinder full of water and touches the bottom. Find to the nearest cubic foot the volume of the water remaining in the cylinder, having given that the radius of the cylinder is 3 ft. and its height 6 ft., the radius of the hemisphere 2 ft., and the height of the cone 4 ft.

5. The diameter of a sphere is 32·4 cm. and its volume is 5·6 times as great as that of a cone whose height is 70·2 cm. Find the radius of the base of the cone to the nearest millimetre.

6. Find the slant surface and volume of the frustum of a pyramid whose thickness is 15 cm., and whose ends are squares on sides of 40 cm. and 24 cm. respectively.

7. How many cubic feet of air are contained in a tent in the form of a circular cylinder surmounted by a cone, the radius of the base being 118 inches, the vertical sides 124·3 inches, and the extreme height to the vertex of the cone 217·9 inches?

8. If a cone of lead 24·6 cm. in height can be hammered into a solid sphere 15·0 cm. in diameter, find to the nearest millimetre the radius of the base of the cone.

9. Find the surface and volume of the solid generated by the revolution of an equilateral triangle about one of its sides, each side being 7·4 cm. in length.

10. A circular room, surmounted by a hemispherical vaulted roof, contains 5236 cubic feet of air, and the internal diameter of the building is equal to the height of the crown of the vault above the floor. Find the height.

11. Prove that the volume of a spherical shell is equal to that of the frustum of a cone whose height is four times the thickness of the shell, and the radii of whose bases are the outer and inner radii of the shell.

12. If V and S are the volume and whole surface of a cone, and V′, S′ the volume and surface of an inscribed sphere, prove that

$$V : V' = S : S'.$$

13. To the ends of a cylinder, whose length is equal to its diameter, are applied hemispheres of the same diameter as the cylinder. If the volume of the whole solid thus formed is 464 cubic centimetres, find its surface.

14. Water passes into a reservoir from a cylindrical pipe 30 cm. in diameter, flowing through the pipe at the rate of 1·25 metre a second. Find how many thousand litres enter the reservoir in 24 hours.

15. Find in kilograms, as nearly as possible with Four-Figure Tables, the weight of sufficient mercury to fill a cylindrical vessel of depth 64 cm., and internal diameter 8 cm. The specific gravity of mercury being taken as 13·6.

16. If in measuring the diameter of a sphere the error on either side may be as much as *one* per cent. of the true diameter, by how much per cent. may the calculated value of the volume exceed the real volume.

17. Find, as nearly as possible with Four-Figure Tables, how many metres of wire 0·4 mm. in diameter can be drawn from 593 kg. of copper, of which the specific gravity is 8·88.

18. Find roughly, using Four-Figure Tables, how many spherical shot, each 2·6 mm. in diameter, can be made from 10·45 kg. of lead, the specific gravity of lead being 11·35.

19. The volume of a frustum of a cone is given by the formula

$$V = \frac{h}{3}(A + \sqrt{AB} + B),$$

h denoting the height of the frustum, and A and B the areas of the two ends. Calculate the volume to the nearest cubic inch, when

$$h = 4{\cdot}5'', \quad A = 28{\cdot}5 \text{ sq. in.}, \quad B = 78{\cdot}6 \text{ sq. in.}$$

What does the formula become (i) when A = B, (ii) when A = 0? Give the geometrical interpretation in each case.

20. A water-tube boiler has 350 tubes of 2·5″ internal diameter, and the length of each tube is 8 feet. Find the total heating surface (*i.e.*, the interior curved surface of the tubes) in square feet to the nearest integer.

21. A right circular cone was measured in such a way that the diameter of the base is known to be between 16·2″ and 16·3″, and the height between 27·5″ and 27·6″. Taking the value of π as 3·1416, find the volume of the cone using (i) the lesser (ii) the greater dimensions. If you give your answer in cubic inches to seven significant figures how many are useless?

22. A cubical block of metal, each edge of which is 36·4 cm is melted down into a sphere. Find the diameter of the sphere as correctly as possible with Four-Figure Tables.

23. The weights of two spheres are in the ratio 8 : 11, and their specific gravities are respectively 1·21 and 0·64. If the diameter of the first is 5·6 cm. find the diameter of the other.

24. A right circular cone is cut by two planes parallel to the base and trisecting the height. Compare the volumes of the three parts into which the cone is divided.

25. Supposing the Earth to be a sphere of diameter 7926 miles, find the length of the Arctic Circle [lat. 66° 30′] as correctly as possible with Four-Figure Tables. Also the area of the zone between latitudes 60° and 65°.

26. Find, as correctly as possible with Four-Figure Tables, the volume of the greatest cube that could be cut from a sphere of diameter 37·62 inches.

27. The water contained in a cubical cistern, of which each internal edge is 6 feet, is found to lose by evaporation 0·04 of its volume in a day. Assuming the loss to arise from evaporation only, find how many ounces of water will be left in the cistern after 10 days.

28. A regular tetrahedron weighs 10·70 kg., its substance being lead of which the specific gravity is 11·35. Find the length of each edge as nearly as possible with Four-Figure Tables.

29. Find the volumes of the following solids of revolution, considering them as the sum or difference of cylinders, cones, or frusta of cones. The solids are generated by the revolution of

(i) an equilateral triangle (side $=a$) about one side ;

(ii) an equilateral triangle (side $=a$) about a line through a vertex parallel to the opposite side ;

(iii) a square (side $=a$) about a line through one vertex parallel to the diagonal which does not pass through that vertex.

(iv) a regular hexagon (side $=a$) about one side.

In each case shew that the volume is equal to that of a prism whose base is the revolving figure, and whose height is the circumference of the circle described by the centre of the figure.

30. Assuming the principle indicated at the end of the last example, find the volume of a solid ring generated by the revolution of a circle of radius $1\frac{3}{4}''$ about a line 7″ from its centre.

ANSWERS TO NUMERICAL EXERCISES.

Since the utmost care cannot ensure absolute accuracy in graphical work, results so obtained are likely to be only approximate. The answers here given are those found by calculation, and being true so far as they go, furnish a standard by which the student may test the correctness of his drawing and measurement. Results within one per cent. of those given in the Answers may usually be considered satisfactory.

Exercises. Page 15.

1. 30°; 126°; 261°; 85°. 11 min.; 37 min.
2. $112\frac{1}{2}$°; 155°; 5 hrs. 45 min.
3. 50°; 8 hrs. 40 min.
4. (i) 145°, 35°, 145°. (ii) 55°, 55°. 86°, 94°.

Exercises. Page 27.

1. 68°, 37°, 75° v. nearly.
2. 6·0 cm.
4. 2·2", 50°, 73° nearly.
5. 37 ft.
6. 101 metres.
7. 27 ft.
8. 424 yds., nearly; N.W.
9. 281 yds., 155 yds., 153 yds.
10. 214 yds.

Exercises. Page 41.

1. 125°, 55°, 125°.
12. 15 secs., 30 secs.

Exercises. Page 43.

3. 21°.
4. 27°.
5. 92°, 46°.
6. 67°, 62°.

Exercises. Page 45.

1. 30°, 60°, 90°.
2. (i) 36°, 72°, 72°; (ii) 20°, 80°, 80°.
3. 40°.
4. 51°, 111°, 18°.
5. (i) 34°; (ii) 107°.
6. 68°.
7. 120°.
8. 36°, 72°, 108°, 144°.
9. 165°.
11. 5, 15.

Exercises. Page 47.

2. (i) 45°; (ii) 36°.
3. (i) 12; (ii) 15.

Exercises. Page 54.

4. (i) 81°; c. (ii) 55°.

10.

Degrees	15°	30°	45°	60°	75°
Cm.	4·1	4·6	5·7	8·0	15·6

11.

Degrees	0°	30°	60°	90°	120°	150°	180°
Cm.	1·0	2·0	3·6	5·0	6·1	6·8	7·0

12. 37 ft.
13. 112 ft.
14. 346 yds. 693 yds.

Exercises. Page 61.

14. 54°, 72°, 54°. 15. 36°. 16. 4.
18. (i) 16; (ii) 45°; (iii) $11\frac{1}{4}$° per sec.

Exercises. Page 68.

2. 6·80 cm. 3. 2·24″. 4. 0·39. 5. 2·54. 8. 10·6 cm.
9. 3·35″. 10. 20 miles; 12·6 km.
11. 147 miles; 235 km. 1 cm. represents 22 km.
12. 1″ represents 15 mi.; 1″ represents 20 mi.

Exercises. Page 79.

3. 0·53 in. 4. 1·3 cm. 5. 2·4″.

Exercises. Page 84.

1. 4·3 cm., 5·2 cm., 6·1 cm. 2. 1·10. 3. 200 yards.
4. 65°, 77 m., 61 m., 56 m. 5. 6·04 knots. S, 15° E, nearly.
6. Results equal. 9 cm. 7. 4·3 cm.; 9·8 cm., 60°; 120°.
8. (i) One solution; (ii) two; (iii) one, right-angled; (iv) impossible.
9. 380 yds. 10. 6·5 cm. 11. 6·9 cm.
12. Two solutions; 10·4 cm. or 4·5 cm. 16. 2·8 cm., 4·5 cm., 5·3 cm
18. 5·8 cm., 4·2 cm. 19. 7 cm., 8 cm.

Exercises. Page 89.

1. 60°, 120°. 2. 3·54″. 3. 2·12″. 4. 4·4 cm
5. 16·4 cm., 3·4 %. 6. 90°. 7. (i) 4·25″; (ii) $B=D=90°$.

Exercises. Page 102.

1. 6 sq. in. 2. 6 sq. in. 3. 2·80 sq. in. 4. 3·50 sq. in.
5. 3·30 sq. in. 6. 3·36 sq. in. 7. 198 sq. m. 8. 42 sq. ft.
9. 10,000 sq. m. 10. 110 sq. ft. 11. 5 cm. 12. 2·6 in.
14. 900 sq. yds.; 48 yds.; 4·8″. 15. 11700 sq. m.
16. 1 cm. = 10 yds. 17. 1·6″. 18. 600 sq. ft. 19. 1152 sq. ft.
20. 100 sq. ft. 21. 156 sq. ft. 22. 110 sq. ft.
23. 288 sq. ft. 24. 72 sq. ft. 25. 75 sq. ft.

Exercises. Page 105.

1. (i) 22 cm.; (ii) 3·6″. 2. 3·4 sq. in. 3. 574·5 sq. in.
4. 1·5″. 5. 1·93″, 75°.

Exercises. Page 107.

1. (i) 180 sq. ft.; (ii) 8·4 sq. in.; 1 hectare
2. (i) 13·44 sq. cm.; (ii) 15·40 sq. cm.; (iii) 20·50 sq. cm.
3. 15 sq. cm.
4. 6·3 sq. in.
5. (i) 8″; (ii) 13 cm.
6. 3·36 sq. in.

Exercises. Page 110.

1. 11400 sq. yds.
2. 6312 sq. m.
3. 2·4 cm.; 5·1 cm.
4. 2·04″; 2·20″.
5.

Angle	0°	30°	60°	90°	120°	150°	180°
Area in sq. cm.	0	7·5	13·0	15·0	13·0	7·5	0

Exercises. Page 111.

1. 66 sq. ft.
2. 84 sq. yds.
3. 126 sq. m.
4. 132 sq. cm.
5. 180 sq. ft.
6. 306 sq. m.

Exercises. Page 113.

1. 6 sq. in.
2. 170 sq. ft.
3. 615 sq. m.
4. 8·4 sq. in.
5. 31·2 sq. cm.
6. 5·20 sq. in.
7. 24 sq. cm.

Exercises. Page 115.

1. (i) 25·5 sq. cm.; (ii) 15·6 sq. cm.
2. (i) 8·95 sq. in.; (ii) 9·5 sq. in.
3. 12500 sq. m.

Exercises. Page 116.

4. 3·3 sq. in.
5. 7·5 cm.
6. 3·6 sq. in.

Exercises. Page 121.

1. (i) 5 cm.; (ii) 6·5 cm.; (iii) 3·7″.
2. (i) 1·6″; (ii) 2·8 cm.
3. 41 ft.
4. 65 miles.
5. 6·1 km.
6. 16 ft.
7. 48 m.
8. 25 miles.
9. 73 m.
10. 62 ft.

Exercises. Page 123.

10. (i) and (iii).
11. 2·83″.
12. 4·24 cm.; 18 sq. cm.
13. 70·71 sq. m.
14. $p=6{\cdot}93$ cm.
16. (i) 20 cm.; 15 cm. (ii) 40 cm.; 39 cm.
17. 35 cm.; 12 cm.; 306 sq. cm.
18. (i) 36 sq. in.; (ii) 90 sq. ft.; (iii) 126 sq. cm.; (iv) 240 sq. yds.
19. 5·1 cm. nearly.

Exercises. Page 127.

1. 7·1 cm. 4. 4·0 cm. 5. 1·6″. 6. 3·1 cm.; 15·6 sq. cm.

Exercises. Page 130.

1. 23·90 sq. cm. 2. 8·40 sq. in.
3. 27·52 sq. cm. 4. 129800 sq. m.

Exercises. Page 134.

3. (i) (8, 5); (ii) (10, 10).
4. (i) (4, 5); (ii) (4, 5); (iii) (−4, −5); (iv) (−4, −5).
5. (6, 5), (12, 10). 6. (5, 8).
7. (i) 17; (ii) 17; (iii) 2·5″; 2·5″.
8. (i) and (ii) 5; (iii) and (iv) 17; (v) and (vi) 37. 9. 10.
14. (0, 0). (7, 5). 15. 13; (9, 6).
16. A straight line passing through the points (4, 0), (0, −4).
17. 117 units of area in each case. 18. A square. 2 sq. in. 1 sq. in.
19. Each = 70 units of area. 20. 9 units of area. 31°, 71°, 78°.
21. (i) 96; (ii) 80; (iii) 120; (iv) 104.
22. (i) 50; (ii) 60; (iii) 120; (iv) 132.
23. Sides 5, 13; area 63. 24. (i) 27; (ii) 21; (iii) 30; (iv) 27·5.
25. (i) 50; (ii) 65·5; (iii) 21; (iv) 83·5.
26. Each side 13; area 120. 27. 13, 10, 15, 8·24, 42, 30.
28. AB = 10, BC = 9, CD = 17, DA = 12·7. Area = 130·5.
29. 10, 13, 5, 5, 3. Area = 60. 30. 160,000 sq. yds. 1000 yds. 320 yds.
31. Side = 15·23; area = 232 units of area.

Exercises. Page 145.

1. 5 cm. 2. 24″. 3. 0·6″, 0·8″. 4. $\sqrt{7}=2{\cdot}6$ cm.
5. 1 ft. 6. 0·6 sq. in. 7. 0·8″.

Exercises. Page 149.

1. 1·7″ 2. $3\sqrt{2}=4{\cdot}2$ cm. 3. $2\sqrt{3}=3{\cdot}5$ cm.
4. 17″. 6. 5 cm.

Exercises. Page 151.

6. 4 cm. 7. 1·3″.

Exercises. Page 153.

2. 1·35″. 3. 1·62″. 5. 0·85″; (2·1″, 2·1″); 2·97″

Exercises. Page 155.

5. 51°. 6. 1·6″; 1·5″, 0·6″.

Exercises. Page 157.

4. (8, 11). 5. 17; 10; (0, −8).

Exercises. Page 161.

1. 74°, 148°, 16°. 2. 115°, 230°. 3. 55°, 8°, 47°.

Exercises. Page 177.

1. 8·0 cm. 2. 0·6″. 3. 8·7 cm. 4. 12°, 67°. 5. 2·5″.

Exercises. Page 179.

3. 3 cm. and 17 cm.

Exercises. Page 181.

1. 72°, 108°, 108°.

Exercises. Page 187.

1. 1·6″. 3. 1·7″. 4. 1·98″, 1·6″.

Exercises. Page 198.

2. 2·3 cm., 4·6 cm., 6·9 cm. 3. 1·39″.
6. 6·9 cm.; 20·78 sq. cm. 7. 3·2 cm.

Exercises. Page 199.

1. 2·12″; 4·50 sq. in. 4. 8·5 cm. 5. 2·0″.

Exercises. Page 200.

4. $128\frac{4}{7}°$; 1·73″.

Exercises. Page 201.

1. 3·46″; 4·00″. 2. 259·8 sq. cm.
4. (i) 41·57 sq. cm.; (ii) 77·25 sq. cm.

Exercises. Page 205.

1. (i) 28·3 cm.; (ii) 628·3 cm. 2. (i) 16·62 sq. in.; (ii) 352·99 sq. in.
3. 11·31 cm.; 10·18 sq. cm. 4. 56 sq. cm. 5. 43·98 sq. in.
7. 30·5 sq. cm. 8. 8·9″. 9. 4″; 3″. 10. 12·57 sq. in.
11. Circumferences, 4·4″, 6·3″. Areas, 1·54 sq. in., 3·14 sq. in.

Exercises. Page 225.

3. 6·4 sq. cm. 4. 3·7″. 5. 10 cm. 6. 1°.

Exercises. Page 228.

1. 630 sq. cm. 15 cm.

Exercises. Page 231.

2. 8·5 cm. 90°.
3. A circle of rad. 6 cm.
4. 5·20″.
6. 0·25″.

Exercises. Page 235.

1. (i) 16 sq. cm. (ii) 16 sq. cm. 2. (i) 16 sq. cm. (ii) 16 sq. cm.
3. 0·8″. 4. (i) 1·2″. (ii) 12·5 cm. 5. (i) 1·6″, 4·0″. (ii) 3·5 cm.
6. Two concentric circles, radii 2 cm. and 6 cm.

Exercises. Page 237.

1. 26″. 2. 48 ft.; 8 ft. 3. 2 cm.; 32 cm.
4. 3·6″. 5. 8100 miles; 10 miles.

Exercises. Page 239.

1. 4 cm. 2. 2·12″. 3. 1·94″. 4. 1·97″.
5. 6·6 cm. 6. 6·6, 2·4. 7. 35·2, 4·8. 8. 3·5 cm.
9. 11·2, 3·2. 10. 9·6, 2·6.

Exercises. Page 241.

1. 2·47″. 2. $-3·24$″

Exercises. Page 245.

1. 8, 2. 2. 7, 7. 3. 9·3, 2·7.
4. 9, -4. 5. 11·32, $-4·32$. 6. 7·24, 2·76.

Exercises. Page 246.

1. 6. 2. 36, 45. 3. 16, 12.
4. (10, $12\frac{1}{2}$); $12\frac{1}{2}$. 5. (17, 18); $12\sqrt{2}=16·97$.
7. 15. 8. 10. 9. Four. (26, 15). 10. 12·84.

Exercises. Page 253.

1. (i) 35; (ii) 8; (iii) a.
3. 4·0″, 5·6″.
4. 16·5 cm., 12·0 cm.
5. 4·0 cm., 2·4 cm.; 16·0 cm., 9·6 cm.

Exercises. Page 258.

1. (i) each $=3:2$; (ii) each $=5:3$; (iii) each $=5:2$.
2. (i) 1·4″; (ii) 0·8″; (iii) 6·4 cm., 2·4 cm.
3. (i) 5·6 cm. (ii) 7·7 cm., 2·8 cm.

Exercises. Page 259.

1. 0·9″, 0·6″; 4·5″, 3·0″; 3 : 2.
2. 2·0 cm., 1·5 cm.; 14·0 cm., 10·5 cm.

Exercises. Page 262.

1. (i) 1·2″; (ii) 2·0″; (iii) 7·7 cm.
2. (i) 2·1″; (ii) 6·3 cm.
3. QB $=3·5$″, BR $=2·5$″.
4. 3·2 cm., 4·2 cm.
5. 2·1″, 1·8″.
6. 5 ft., $12\frac{3}{4}$ ft., $9\frac{1}{4}$ ft.
7. 1·2″, 1·3″, 1·95″.
8. $5\frac{1}{3}$ cm.
9. 0·8 cm., 1·4 cm., 2·1 cm.

Exercises. Page 271.

1. 17, $\frac{8}{17}$, $\frac{15}{17}$, $\frac{8}{15}$.
2. 37. sine $=\frac{12}{37}$, cosine $=\frac{35}{37}$, tangent $=\frac{12}{35}$.
4. $\frac{12}{13}$, $\frac{5}{12}$, $\frac{77}{85}$, $\frac{77}{36}$.
6. 37°.
7. 35°, 26°, 45°.
8. A $=58°$, cos A $=0·53$.
10. AC $=7·8$, A $=39°$, sin A $=0·63$, cos A $=0·78$.
11. 3·4, 35°.

Exercises. Page 278.

1. (i) 1·0″; (ii) 0·9″; (iii) 6·0 cm.
2. 1·4″, 0·6″; 3·5″, 1·5″.
3. (i) 2·0; (ii) 2·8; (iii) 20.
4. 1·6 cm., 2·4 cm., 3·2 cm.
5. 1·8″, 1·2″, 0·9″.
6. 2·7″.
7. (i) 1·73; (ii) 3·16; (iii) 1·67.
8. (i) 3; (ii) 3·21; (iii) 2·28.
9. (i) 1·2″, 1·6″, 2·0″; (ii) 3·0 cm., 3·6 cm., 4·5 cm. (iii) 2·5 cm., 4·3 cm., 5·0 cm. (iv) $b=3·4$″, $c=2·1$″, nearly.

Exercises. Page 279.

1. 140 m., 160 m.; 125 m.
2. $12\frac{1}{2}$ yds.
3. $42\frac{1}{2}$ miles.
4. 30 ft., 4 ft.
5. 24 ft., 2 ft. 4 in.
6. 60 ft.
7. 72 ft.
8. 106 ft.

Exercises. Page 284.

3. 0·52. 5. 31 : 28, nearly.

Exercises. Page 287.

1. 10·5 sq. in. 2. 3·0 cm. 3. 64 sq. cm.
4. 11·0″. 5. 33·9 acres.

Exercises. Page 289.

6. 4·9 cm. 7. 8·0 cm.

Exercises. Page 291.

1. $\frac{1}{9}$. 2. 20 sq. ft. 3. 10 sq. cm. 4. 7 : 5. 5. 5·6″.

Exercises. Page 294.

2. 3·46″, 4·33″, 5·54″. 3. 9 ft. 3 in. 4. 3·75 sq. cm.
5. 4·5 cm. 6. 15·48 sq. in. 7. 3·6 m. 1·5 m.
8. 90 acres. 9. 512 acres. 10. 1 cm. represents 15 metres.

Exercises. Page 295.

2. $1 : \sqrt{2}$. 6. $\frac{1}{64}$. 7. 256 : 81.

Exercises. Page 297.

3. 2·5 sq. cm., 6·4 sq. cm. 4. 4 : 1.
5. 7·2″. 8. 6·2 cm., 3·8 cm.

Exercises. Page 299.

1. $1 : \sqrt{2}$. 3. 4·6 cm. 4. 6·9 cm.

Exercises. Page 301.

1. 11·56 sq. in. 2. (i) 100, (ii) 576 units of area ; (iii) 18·5.
3. (1·3, 2·4), (1·3, –2·4) ; 4·8 ; 2·4, nearly.

Exercises. Page 305.

10. (i) $10\frac{5}{8}''$ (ii) $15\frac{5}{8}$ ft.

Exercises. Page 313.

1. 15 ; (8, 6). 2. 15, 3·75 ; (0, 20), (0, 5). 3. 0·33″.
4. 1·17″. 6. 0·55″. 7. 0·93″.

Exercises. Page 317.

1. 15·3 sq. cm.
2. 8·55″.
3. 43·30 sq. cm.
4. 90°.
8. 6·83″; 1·17″.
10. 20·78 sq. cm., 20·8 cm.
12. 56°.
14. 1·2″; $22\frac{1}{2}$°.
15. 63 ft.
16. 18 sq. cm.
17. $60\frac{1}{2}$ sq. cm.

Exercises. Page 322.

1.

a	15°	30°	45°	60°	75°
PR	8·3	9·2	11·3	16·0	30·9

17·6 cm.; 25°.

2. (i) 8·9 sq. cm.; (ii) 72° or 108°; (iii) at right angles.
3. When the rod is equally inclined to the rulers.
4. When P is the mid-point of AB (i) is a maximum, (ii) is a minimum.
5. Minimum when $a=3$.
6. 45°.
7. 3·4″.
8. 9.

Exercises. Page 326.

2. (i) 6·0″; (ii) 1·2 cm.

Exercises. Page 329.

1. 4·0 cm., 8·8 cm.
2. (i) 1·20″; (ii) 1·98″.

Exercises. Page 355.

4. 7·0 cm.
5. 3·0″; 2·5″.
6. (ii) 42·4 cm.; (iii) 56·6 cm.

Exercises. Page 361.

2. 5·0″.
6. 0·324.

Exercises. Page 362.

4. 10·4 cm. 3·[illegible] cm. 10·2 cm. 0·385, 0·339.

Exercises. Page 375.

4. (i) 0·8000; (ii) 0·8571.
5. 0·2800.

Exercises. Page 382.

1. 3, 4, 5.
3. 13.

Exercises. Page 389.

1. 72 sq. ft.
2. 17 cm.; $\frac{15}{17}$; 144·48 sq. cm.
3. 10 cm.; 55·54 sq. cm.

Exercises. Page 391.

1. 25·50 m.
2. 650 litres; 520 kg.
3. $6\frac{1}{2}$ millions.
4. 1188 kg.

Exercises. Page 393.

1. 196·3.
2. 20400 kg.
3. 74·88 kg.
4. 5445.
5. 8s. 4*d*.
6. 21·42 kg.
7. 5s. 11*d*.
8. 65 cm.; 65 cm.
9. 1 cm.
10. 12 cm., 15 cm., 18 cm.
11. 7 cm., 6 cm., 5 cm.
12. 5·8 cm., 200 sq. cm., 192·4 cu. cm.
13. 12 cm., 4 cm.
14. 29 cm.
15. 14·4".
16. 720 cu. cm., 480 sq. cm.
17. 360 cu. cm., 432 sq. cm.
18. 12,000 cu. cm.
19. 4·6 cu. ft.
20. 375,000 gall., 167 tons.
21. 23661.
22. 65 hrs. $6\frac{1}{4}$ mins.
23. (i) 3 : 4; (ii) 717 : 1000.
24. 459 working days.

Exercises. Page 401.

2. (i) 91·38 sq. in.; (ii) 84 cu. in.
3. (i) 308 cu. cm.; (ii) 280 cu. cm.
4. (i) 9·85"; (ii) 10·63".
5. (i) 223·6 sq. cm.; (ii) 333·3 cu. cm.
6. 8 cm., 1152 cu. cm.
7. (i) 0·28; (ii) 0·49 sq. in.
8. (i) 5·8 cm.; (ii) 28·87 sq. cm.; (iii) $\frac{1}{3}$. 60°.
9. 8·7 cm.

Exercises. Page 409.

1. 377·400 cu. cm. 15·612, 16·050 ; 4·31%, 4·08%.
2. $87\frac{1}{2}$ lbs.
3. 36° 52′.
4. (i) 104·6 sq. cm. ; (ii) 110·8 cu. cm.
5. 4·2 cm., 50·9 cu. cm.
6. 384 units of area.
7. 162·4 cu. cm.
12. 70° 32′.

Exercises. Page 413.

1. (i) 151 sq. cm. ; 226 cu. cm. (ii) 204 sq. cm. ; 458 cu. cm.
2. 528 sq. cm.
3. 122 cu. cm.
4. 123 sq. cm.
5. 528 sq. cm.
6. 10·20 m.
7. 0·006″.
8. 5000 cu. cm. ; 15·9 cm.
9. 7 cu. ft.
10. 88·1 kg.
11. 18·85 m. ; 525·9 gr.

Exercises. Page 417.

2. (i) 188 sq. cm. ; 302 cu. cm. (ii) 14 sq. cm. ; 5 cu. cm.
3. 1414 sq. cm.
4. 27 cu. cm.

Exercises. Page 421.

1. 816 sq. cm.
2. 110 sq. cm.
3. 148 cu. cm.
4. 79 sq. cm. ; 88 cu. cm.
5. 129·2 sq. cm.
6. 140 cu. cm.
9. $\dfrac{n^3-(n-1)^3}{n^3}$.
10. 10 cm., 5 cm.

Exercises. Page 422.

1. 20 cu. cm.
2. 268 yds. 1 ft.
3. 5·4 m.
4. 1017 sq. cm.
5. 51 min. 12 secs.
6. 1 : 2.
7. 18 sq. cm.
8. 376·99 sq. cm.
9. 23 : 39.
10. 390 sq. cm.

Exercises. Page 432.

1. (i) 72 sq. cm. ; 58 cu. cm. (ii) 1385 sq. cm. ; 1849 cu. cm.
2. £16. 19s. 4d.
3. 0·7 cm.
4. 15.
5. 381 cu. cm.
6. 286 cu. cm.
7. 1 cm.
8. 418 sq. cm. ; 1·694 kg.
9. 667 cu. cm.
10. 4·2 cm.
11. 1 in.
12. 1 cm.
13. 8 : 17.
14. 152 kg.
15. 5·654 kg.
16. 24300 m. ; 10·8%.
17. 1922·66 sq. cm. ; 7125·15 cu. cm.
18. 439·82 sq. cm. ; 318·35 cu. cm.
19. 571·77 sq. cm.
20. 611·3 sq. ft.
21. 42 ft.

Exercises. Page 435.

2. (i) 24880 mi. ; (ii) 1·152 mi. ; (iii) 14270 mi. ; (iv) 645 mi.
5. (i) 197,100,000 sq. mi. ; (ii) 8,172,000 sq. mi. ; (iii) 78,590,000 sq. mi.

Exercises. Page 440.

1. 31 : 50.
2. 23 cm.
3. 204 sq. cm. ; 332 cu. cm.
4. 136 cu. ft.
5. 6·6 cm.
6. 2803 sq. cm. ; 15680 cu. cm.
7. 3937 cu. ft.
8. 8·3 cm.
9. 298 sq. cm. ; 318 cu. cm.
10. 20 ft.
13. 315 sq. cm.
14. 7634000.
15. 43·75 kg.
16. 3·0% nearly.
17. 531,500 m.
18. 100,000.
19. 232 cu. in.
20. 1833 sq. ft.
21. 1889 cu. in. 1920 cu. in. Data suffice for 4 significant digits only.
22. 45·16 cm.
23. 7·7 cm.
24. 1 : 7 : 19.
25. 9926 mi. ; 3,976,000 sq. mi.
26. 10250 cu. in.
27. 143700 oz.
28. 20·00 cm.
29. (i) $\frac{\pi a^3}{4}$; (ii) $\frac{1}{3}\pi a^3$; (iii) $\pi a^3\sqrt{2}$; (iv) $\frac{9}{2}\pi a^3$.
30. 423·16 cu. in.

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